SURGERY OBSTRUCTIONS FROM KHOVANOV HOMOLOGY

LIAM WATSON

Abstract. For a 3-manifold with torus boundary admitting an appropriate involution, we show that Khovanov homology provides obstructions to certain exceptional Dehn fillings. For example, given a strongly invertible knot in S^3, we give obstructions to lens space surgeries, as well as obstructions to surgeries with finite fundamental group. These obstructions are based on homological width in Khovanov homology, and in the case of finite fundamental group depend on a calculation of the homological width for a family of Montesinos links. Precisely, if a link has homological width greater than 2 then the two-fold branched cover must have infinite fundamental group. As an illustration, we recover the fact that the figure eight knot does not admit finite fillings, and prove that a family of pretzel knots, including the $(-2,5,5)$-pretzel, do not admit finite fillings. Further examples are also explored in order to compare the obstructions given here with other surgery obstructions, in particular those arising from Heegaard-Floer homology. We also illustrate these obstructions on knots in manifolds other than S^3 by example, studying surgery on a knot in the Poincaré homology sphere. Finally, a characterization of the trivial knot – among strongly invertible knots – from Khovanov homology is given.

1. Introduction

In his pioneering work on the geometry and topology of 3-manifolds, Thurston showed that a hyperbolic manifold M with torus boundary admits a finite number of exceptional Dehn fillings [53, 54]. That is, those closed manifolds obtained from M by attaching a solid torus to the boundary that are non-hyperbolic. Since then, the question of understanding and classifying exceptional surgeries has received considerable attention (see surveys by Gordon [16] and Boyer [8]).

Perhaps the simplest non-hyperbolic manifold is a lens space. Restricting to complements of knots in S^3, Moser [35] showed that torus knots always admit lens space surgeries. Subsequently, Bailey and Rolfsen [1] constructed an example of a lens space surgery on a non-torus knot (a particular cable of the trefoil), and Fintushel and Stern [12] obtained further examples, including hyperbolic knots, that admit lens space surgeries.

In [4] Berge gives a list of knots in S^3 that admit lens space surgeries. These knots are referred to as Berge knots, and it has since been conjectured by Gordon that this list is complete. That is, if a knot in S^3 admits a lens space surgery then it must be a Berge knot; this has become known as the Berge conjecture. Since Berge knots are genus 2, they are strongly invertible by a result of Osborne [36]. As a result, the Berge conjecture may be restated in two steps: first show that any

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knot admitting a lens space surgery is strongly invertible, then show that a strongly invertible knot admitting a lens space surgery is a Berge knot.

The complement of a strongly invertible knot admits a tangle associated to the quotient of the strong inversion. This *associated quotient tangle* (see Definition 3.4) can be a useful object when studying surgery on the knot in question. Indeed, surgeries on the strongly invertible knot correspond in a natural way to closures of the associated quotient tangle by a rational tangle. In this way, the manifold obtained by surgery is naturally a two-fold branched cover of S^3, branched over an appropriate closure of the associated quotient tangle. While both parts of the restatement of the Berge conjecture given above remain open, in light of the second it may be useful to use invariants of knots in S^3 to study the branch sets associated to such surgeries.

In this setting Khovanov homology [24] may be used to give information about the manifold obtained via surgery in the cover:

Theorem 5.6. *Let T be the associated quotient tangle of a strongly invertible knot K in S^3. If T has the property that the links obtained from it by attaching rational tangles have thick Khovanov homology, then K does not admit a non-trivial lens space surgery. Moreover, under mild hypothesis on T it suffices to verify only a finite number of branch sets associated to integer surgeries to obtain the conclusion for all possible fillings.*

This depends on a result that we attribute to Hodgson and Rubinstein [20] and Lee [29]: lens spaces arise only as the branched covers of homologically thin links (see Theorem 5.1). As a result of a particular stable behaviour of Khovanov homology for branch sets associated to such surgeries (see Lemma 4.10), it suffices to calculate a finite collection of Khovanov homology groups to apply this obstruction. Indeed, we will see in application that often a single Khovanov homology group suffices and that the required genericity hypothesis alluded to on T (see Definition 5.4) are easily satisfied.

Recently, the question of lens space surgeries has been treated by Ozsváth and Szabó [40] and Rasmussen [44] from the point of view of Heegaard-Floer homology. Indeed, some progress on the Berge conjecture has been made by way of Heegaard-Floer homology (see the programs put forth by Baker, Grigsby and Hedden [2, 17] and Rasmussen [46]). Moreover, Ozsváth and Szabó [41] have shown that there is a close relationship (by way of a spectral sequence) between the Khovanov homology of a link and the Heegaard-Floer homology of the two-fold branched cover of S^3, branched over the the link. From this point of view, it is natural to ask how the obstructions from the two theories might be related.

The work of Ozsváth and Szabó [40] gives, more generally, obstructions to L-space surgeries. Interesting examples of L-spaces include two-fold branched covers of links with thin Khovanov homology (see [41] and Proposition 4.2), as well as manifolds admitting elliptic geometry [40]. In a similar vein, it can be shown that the branch set associated to a manifold with finite fundamental group (viewed as a two-fold branched cover of S^3) has relatively simple Khovanov homology.

Theorem 5.2. *If the two fold branched cover of L has finite fundamental group, then the reduced Khovanov homology of L is supported in at most 2 diagonals.*

As a result, we obtain the following:

Theorem 5.7. *Let T be the associated quotient tangle of a strongly invertible knot K in S^3. If T has the property that the links obtained from it by attaching rational tangles have reduced Khovanov homology supported in more than two diagonals, then K does not admit a non-trivial surgery with finite fundamental group. Moreover, under mild hypothesis on T it suffices to verify only a finite number of branch sets associated to integer surgeries to obtain the conclusion for all possible fillings.*

One may view the obstructions given above as a rough correspondence between the geometry of a two-fold branched cover and the Khovanov homology of the branch set: simple manifolds (in terms of geometry) tend to have simple branch sets (in terms of Khovanov homology). That such a correspondence exists is interesting in light of the fact that Khovanov homology, while relatively strong as an invariant of knots in S^3 and beautiful in its own right, lacks a complete geometric interpretation. As a result, this invariant has seen relatively few geometric applications despite receiving much attention since its inception. That said, the applications that have arisen have been particularly interesting. Perhaps most notable, Rasmussen [43] obtained a combinatorial proof of the Milnor conjecture using Khovanov homology. As such, the search for further applications of the theory is of central interest.

It should pointed out that the obstructions given here apply to a wider class of manifolds with torus boundary than complements of knots in S^3: any manifold that has the structure of a two-fold branched cover may be studied by way of the Khovanov homology of the branch set. For example, Tange [51] has studied knots in the Poincaré homology sphere and shown that many admit lens space space surgeries. Although we do not pursue it here beyond a single example (see Section 6.5), since this manifold arises naturally as a two-fold branched cover we can apply the techniques of the present paper to surgery questions of this nature whenever the knot in question is strongly invertible.

Organization of the paper. Section 2 gives a review of Khovanov homology, and in particular the skein exact sequence, as the grading conventions used in this work are non-standard and adapted to the study of homological width. We prove a degenerate case of a version of the skein exact sequence due to Manolescu and Ozsváth [31] (compare Proposition 2.6 and Proposition 2.7), and introduce the σ-normalized Khovanov homology. This is a useful $\mathbb{Z}$-graded object in the context of this work, and seems to be a natural and interesting object in its own right.

In Section 3 we review the required elements of surgery theory on 3-manifolds, and introduce the notion of a simple, strongly invertible knot manifold (Definition 3.3) and the associated quotient tangle (Definition 3.4). This is precisely the family of manifolds for which fillings may be studied by way of Khovanov homology.

Section 4 establishes the stability of Khovanov homology for branch sets associated to integer surgery (see Lemma 4.10). As a result, we may define the maximal and minimal width of the Khovanov homology for the branch set of an integer filling (Definition 4.16), and this is used to give width bounds for the branch set of an arbitrary filling in terms of the branch sets associated to integer fillings. We also discuss quasi-alternating links as they arise naturally in this context (see Theorem 4.7), and in particular we demonstrate that L-spaces arising from large surgery on a Berge

knot may be realized as the two-fold branched cover of a quasi-alternating link (see Proposition 4.8).

Section 5 applies the above material to the main results of this paper. We prove upper bounds for the width of the Khovanov homology of a branch set associated to a lens space surgery (Theorem 5.1), as well as for the width of a finite filling (Theorem 5.2). These bounds, combined with the stable behaviour of the associated Khovanov homology groups developed in Section 4 give rise to our main results on surgery obstructions (Theorem 5.6 and Theorem 5.7).

Finally, we turn to examples and applications in Section 6. As a first example, we recover the fact that the figure eight does not admit finite fillings. We also show that surgery on the $(-2,p,p)$-pretzel knot does not yield a manifold with finite fundamental group (for $p \in \{5,7,\dots,31\}$). Finite fillings on this family of Montesinos links was left unresolved[1] in Mattman's extensive study of the problem using character variety methods [32]. We attempt to contrast our obstructions with the powerful obstructions of Heegaard-Floer homology by exhibiting that the knot 14^n_{11893} (see Figure 16) does not admit surgeries with finite fundamental group. Finally, we give characterization of the trivial knot (among strongly invertible knots) in terms of Khovanov homology as an application of homological width. This shows that, combined with knowledge of the symmetry group of a knot, Khovanov homology may be used to detect the trivial knot. In light of the connection between Khovanov homology and Heegaard-Floer homology for two-fold branched covers, it is interesting to compare this with the fact that knot Floer homology detects the trivial knot [38]. We also study surgery on a knot in the Poincaré sphere, as illustration of these techniques beyond knots in S^3.

On conventions and calculations. Throughout we will use Rolfsen's notation [47] for knots with 10 or fewer crossings, and `Knotscape` notation [21] for knots with more that 10 crossings, as has become standard in the literature. Calculations of the Khovanov homology groups given in this work were performed using the program `JavaKh` by Bar-Natan and Green [3].

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2. Khovanov homology

We begin by reviewing Khovanov homology [24] to fix notation and conventions. We work with the reduced version of the theory with coefficients in the field $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. For definitions see Khovanov's work [25], as well as work of Shumakovitch [50]. To a link $L \hookrightarrow S^3$, this theory associates a bi-graded group (or, $\mathbb{F}$-vector space) $\widetilde{\mathrm{Kh}}(L)$ with primary grading δ and secondary grading q. This grading is non-standard: to recover the original conventions, the primary grading is given by $\delta + 2q$, and secondary grading by $2q$.

[1] Mattman's classification has very recently been completed by Ichihara and Jong applying Heegaard-Floer homology techniques [22], and independently treated by Futer, Ishikawa, Kabaya, Mattman and Shimokawa [13].

Unless otherwise specified, we will consider $\widetilde{\mathrm{Kh}}(L)$ as a *relatively* $\mathbb{Z}\oplus\mathbb{Z}$-graded group. This has the advantage that $\widetilde{\mathrm{Kh}}(L)$ becomes an invariant of unoriented links, resulting from the fact that the absolute grading in Khovanov homology is obtained by applying an overall shift to the cube of resolutions defining the chain complex [24].

As a relatively graded group, this homology theory categorifies to the Jones polynomial [23] in the following sense:

Theorem 2.1 (Khovanov [25])**.** *Let $u=\delta+q$. Then there is a unique absolute $\mathbb{Z}\oplus\frac{1}{2}\mathbb{Z}$-grading (in (u,q)) on $\widetilde{\mathrm{Kh}}(L)$ with the property that*

$$V_L(t)=\textstyle\sum_{u,q}(-1)^u t^q \operatorname{rk}\widetilde{\mathrm{Kh}}^u_q(L),$$

where $V_L(t)\in\mathbb{Z}[t^{\frac{1}{2}},t^{-\frac{1}{2}}]$ is the Jones polynomial.

We remark that the universal coefficient theorem ensures that the graded Euler characteristic (giving rise to the Jones polynomial) is invariant of the coefficient field. Note also that taking $u=\delta+q$ as in the above theorem without fixing an absolute grading recovers the Kauffman bracket of the underlying link.

According to our grading conventions, the usual Euler characteristic

$$\chi(\widetilde{\mathrm{Kh}}(L))=\textstyle\sum_\delta(-1)^\delta \operatorname{rk}\widetilde{\mathrm{Kh}}^\delta(L)$$

is obtained by collapsing the q grading. Note that this is only well defined up to sign as δ is a relative integer grading; we fix the convention $\chi\geq 0$. Recall that $\det(L)=|H_1(\mathbf{\Sigma}(S^3,L);\mathbb{Z})|$ where $\mathbf{\Sigma}(S^3,L)$ is the two-fold branched cover of S^3, branched over L.

Proposition 2.2. *With the above notation and conventions, $\chi(\widetilde{\mathrm{Kh}}(L))=\det(L)$.*

Proof. By definition,

$$\begin{aligned}\chi(\widetilde{\mathrm{Kh}}(L)) &= \big|\textstyle\sum_{\delta,q}(-1)^\delta \operatorname{rk}\widetilde{\mathrm{Kh}}^\delta_q(L)\big|\\ &= \big|\textstyle\sum_{u,q}(-1)^{u-q} \operatorname{rk}\widetilde{\mathrm{Kh}}^u_q(L)\big|\\ &= \big|\textstyle\sum_{u,q}(-1)^{u}(-1)^q \operatorname{rk}\widetilde{\mathrm{Kh}}^u_q(L)\big|\\ &= |V_L(-1)|,\end{aligned}$$

and the result follows from the well known identity $\det(L)=|V_L(-1)|$. □

Forgetting the q-grading in this way yields

$$\widetilde{\mathrm{Kh}}(L)\cong\mathbb{F}^{b_1}\oplus\cdots\oplus\mathbb{F}^{b_k}=\textstyle\bigoplus_{\delta=1}^k\mathbb{F}^{b_\delta}$$

for non-negative integers b_i, where b_1 and b_k are positive. As a result, we arrive naturally at the following:

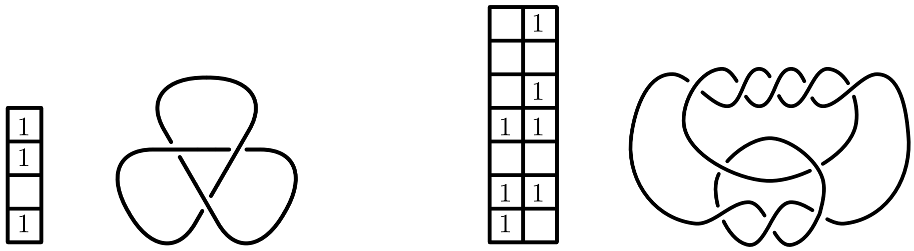


FIGURE 1. The reduced Khovanov homology of the trefoil (left) with $w = 1$, and the knot 10_{124} (right) with $w = 2$. The primary relative grading (δ) is read horizontally, and the secondary relative grading (q) is read vertically. The values at a given bi-grading give the ranks of the abelian group (or $\mathbb{F}$-vector space) at that location; trivial groups are left blank.

Definition 2.3. *The homological width of L is given by $w(L) = k$, the number of δ-gradings supporting the reduced Khovanov homology. Links for which $w = 1$ are called thin (or homologically thin), while links with $w > 1$ are termed thick (or homologically thick).*

Hence, our grading convention gives homological grading by *diagonals* of slope 2 from the standard bi-grading in Khovanov homology. A result due to Lee shows that non-split alternating links give a family of thin links.

With these conventions in hand, $\chi(\widetilde{\mathrm{Kh}}(L)) = |\sum_{i=1}^{k}(-1)^{\delta}b_{\delta}|$ and $\mathrm{rk}\,\widetilde{\mathrm{Kh}}(L) = \sum_{\delta=1}^{k} b_{\delta}$, giving rise to examples of thick links.

Proposition 2.4. *Any link L with $\det(L) = 0$ must have $w(L) > 1$.*

Proof. Since $V_L(t)$ is a non-zero polynomial [23, Theorem 15], it follows that $\mathrm{rk}\,\widetilde{\mathrm{Kh}}(L) > 0$ for any link L. In particular there is at least one $b_{\delta} \neq 0$. Therefore if $\det(L) = \chi\big(\widetilde{\mathrm{Kh}}(L)\big) = 0$ there must be at least two such gradings supporting non-trivial groups. $\square$

2.1. **Mapping cones.** The skein exact sequence for reduced Khovanov homology relates the homology of a link with a fixed crossing $\times$ to the homology of the links obtained from the 0-resolution $\asymp$ and the 1-resolution $)($. For a link $L(\times)$ with distinguished positive crossing we have that

$$\longrightarrow \widetilde{\mathrm{Kh}}\left(L()()\right)[-\tfrac{1}{2}c, \tfrac{1}{2}(3c+2)] \longrightarrow \widetilde{\mathrm{Kh}}\left(L(\times)\right) \longrightarrow \widetilde{\mathrm{Kh}}\left(L(\asymp)\right)[-\tfrac{1}{2}, \tfrac{1}{2}] \longrightarrow$$

Here, $[\cdot,\cdot]$ is a shift in the bi-grading via $\widetilde{\mathrm{Kh}}(L)[i,j]^{\delta}_{q} = \widetilde{\mathrm{Kh}}^{\delta-i}_{q-j}(L)$, and $c = n_-(L()()) - n_-(L(\times))$, the difference in the number of negative crossings for some choice of orientation on the affected components of the resolution $L()()$ (these grading conventions are consistent with [31, 45]). Note that the connecting homomorphism raises the primary grading by 1. Similarly, for a link with distinguished negative crossing $L(\times)$ we have

$$\longrightarrow \widetilde{\mathrm{Kh}}\left(L(\asymp)\right)[\tfrac{1}{2}, -\tfrac{1}{2}] \longrightarrow \widetilde{\mathrm{Kh}}\left(L(\times)\right) \longrightarrow \widetilde{\mathrm{Kh}}\left(L()()\right)[-\tfrac{1}{2}(c+1), \tfrac{1}{2}(3c+1)] \longrightarrow$$

Omitting grading shifts for the moment, and simplifying with the notation $\times$ for $L(\times)$, these exact sequences are often represented by exact triangles of the form

$$\begin{tikzcd}[column sep=small] & \widetilde{\mathrm{Kh}}(\times) \arrow[dr] & \\ \widetilde{\mathrm{Kh}}()\,() \arrow[ur] & & \widetilde{\mathrm{Kh}}(\asymp) \arrow[ll, dashed, "{[1,0]}"'] \end{tikzcd}$$

Since we are working over a field, the homology $\widetilde{\mathrm{Kh}}(L)$ is completely determined by the groups $\widetilde{\mathrm{Kh}}(\asymp)$ and $\widetilde{\mathrm{Kh}}()\,()$, together with the connecting homomorphism. This leads directly to the notion of a mapping cone (see [41, Section 4], for example), which is a useful point of view in the present setting. That is, we have

$$\widetilde{\mathrm{Kh}}(\times) \cong H_*\left(\widetilde{\mathrm{Kh}}(\asymp) \to \widetilde{\mathrm{Kh}}()\,()\right)$$

where the connecting homomorphism raises homological δ-grading by one as above.

Replacing the grading shifts, we have

$$\begin{aligned} \widetilde{\mathrm{Kh}}(\times_+) &\cong H_*\left(\widetilde{\mathrm{Kh}}(\asymp)[-\tfrac{1}{2},\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}()\,()[-\tfrac{1}{2}c,\tfrac{1}{2}(3c+2)]\right) \\ \widetilde{\mathrm{Kh}}(\times_-) &\cong H_*\left(\widetilde{\mathrm{Kh}}()\,()[-\tfrac{1}{2}(c+1),\tfrac{1}{2}(3c+1)] \to \widetilde{\mathrm{Kh}}(\asymp)[\tfrac{1}{2},-\tfrac{1}{2}]\right) \end{aligned}$$

The singly δ-graded group will be useful in many instances, and in this setting the mapping cones simplify to yield

$$\begin{aligned} \widetilde{\mathrm{Kh}}(\times_+) &\cong H_*\left(\widetilde{\mathrm{Kh}}(\asymp)[-\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}()\,()[-\tfrac{1}{2}c]\right) \\ \widetilde{\mathrm{Kh}}(\times_-) &\cong H_*\left(\widetilde{\mathrm{Kh}}()\,()[-\tfrac{1}{2}(c+1)] \to \widetilde{\mathrm{Kh}}(\asymp)[\tfrac{1}{2}]\right) \end{aligned}$$

where $[\cdot]$ shifts the δ-grading.

2.2. Normalization and Support.

2.2. Normalization and Support. In calculations involving the skein exact sequence absolute gradings are essential. Therefore, we will generally need to fix an orientation, although the final result (as a relatively graded group) will not depend on this choice.

In particular, $w(L)$ depends only on $\widetilde{\mathrm{Kh}}(L)$ as a relatively graded group, however determining this quantity in practice will depend on absolute gradings. For this reason we introduce the notion of support, denoted by $\mathrm{Supp}(\widetilde{\mathrm{Kh}}(L))$, as an absolutely $\mathbb{Z}$-graded quantity. Thus if

$$\widetilde{\mathrm{Kh}}(\times_+) \cong H_*\left(\widetilde{\mathrm{Kh}}(\asymp)[-\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}()\,()[-\tfrac{1}{2}c]\right)$$

and $\mathrm{Supp}\left(\widetilde{\mathrm{Kh}}()\,()[-\tfrac{1}{2}c]\right) \subseteq \mathrm{Supp}\left(\widetilde{\mathrm{Kh}}(\asymp)[-\tfrac{1}{2}]\right)$ the we may write

$$\widetilde{\mathrm{Kh}}(\times_+) \cong H_*\left(\begin{tikzcd}[column sep=small] \mathbb{F}^{b_1} \arrow[dr] & \mathbb{F}^{b_2} \arrow[dr] & \cdots \arrow[dr] & \mathbb{F}^{b_k} \\ \mathbb{F}^{b'_1} & \mathbb{F}^{b'_2} & \cdots & \mathbb{F}^{b'_k} \end{tikzcd}\right)$$

for $b_i, b'_i \geq 0$, since the connecting homomorphism raises δ-grading by 1.

The following will be a useful absolutely $\mathbb{Z}$-graded object:

Definition 2.5. *The σ-normalized Khovanov homology is an absolutely $\mathbb{Z}$-graded theory defined by $\widetilde{\mathrm{Kh}}_\sigma(L) = \widetilde{\mathrm{Kh}}(L)[-\frac{\sigma(L)}{2}]$, where $\sigma(L)$ denotes the signature of the link L.*

This turns out to be a natural absolute grading to consider, despite the fact that we are interested only in the relative grading, ultimately. Of course, $\widetilde{\mathrm{Kh}}_\sigma(L)$ and $\widetilde{\mathrm{Kh}}(L)$ coincide as relatively $\mathbb{Z}$-graded groups.

2.3. **The Manolescu-Ozsváth exact sequence.** As a singly graded theory, there is a useful special case in which the skein exact sequence simplifies nicely in terms of the σ-normalization.

Proposition 2.6 (Manolescu-Ozsváth [31, Proposition 5])**.** *Let $L = L(\times)$ be a link with some distinguished crossing, and set $L_0 = L(\asymp)$ and $L_1 = L()\,()$. If $\det(L_0), \det(L_1) > 0$ and $\det(L) = \det(L_0) + \det(L_1)$ then*

$$\widetilde{\mathrm{Kh}}_\sigma(L) = H_*\left(\widetilde{\mathrm{Kh}}_\sigma(L_0) \to \widetilde{\mathrm{Kh}}_\sigma(L_1)\right).$$

In the standard notation, this takes the form

$$\widetilde{\mathrm{Kh}}(L)[-\tfrac{\sigma}{2}] = H_*\left(\widetilde{\mathrm{Kh}}(L_0)[-\tfrac{\sigma_0}{2}] \to \widetilde{\mathrm{Kh}}(L_1)[-\tfrac{\sigma_1}{2}]\right).$$

where $\sigma = \sigma(L)$, $\sigma_0 = \sigma(L_0)$ and $\sigma_1 = \sigma(L_1)$ (see [31]). Notice that in this setting the orientation of the resolved crossing does not play a role so that a single expression replaces the pair of exact sequences.

2.4. **On the signature of a link.** We briefly review the work of Gordon and Litherland, constructing the signature of a link via the Goeritz matrix [15]. The conventions we adopt are that of Manolescu and Ozsváth [31], since our interest will be in proving a degenerate form of Proposition 2.6.

The complement of a projection of a link L is divided into regions that may be coloured black and white in an alternating fashion to obtain the checkerboard colouring. Denote the white regions by $R_0, R_1, \ldots, R_n$. We may assume that every crossing c of the diagram for L is incident to distinct white regions, and assign an incidence number $\mu(c)$ and type by the conventions of Figure 2.

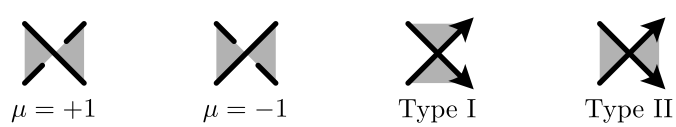


FIGURE 2. Incidence numbers and crossing types.

The incidence number of the diagram for L is obtained by taking the sum of incidences over crossings of type II. Setting

$$\mu(L) = \sum_{c \text{ of type II}} \mu(c),$$

the Goeritz matrix of G for the diagram of L is the $n \times n$ symmetric matrix

$$g_{ij} = \begin{cases} -\sum_{c \in R_{ij}} \mu(c) & i \neq j \\ -\sum_{i \neq k} g_{ik} & i = j \end{cases}$$

where $R_{ij} = \overline{R_i} \cap \overline{R_j}$ for $i, j \in \{1, \dots, n\}$.

From the work of Gordon and Litherland [15], the signature of the link L is given by $\sigma(L) = \mathrm{signature}(G) - \mu(L)$ and $\det(L) = |\det(G)|$.

2.5. **Degenerations.** We now prove a degeneration of Manolescu and Ozsváth's exact sequence when one of determinants of the pair of resolutions vanishes. Once again, a single expression is obtained in each case.

Proposition 2.7. *Using the same conventions as Proposition 2.6, if* $\det(L_0) = 0$ *and* $\det(L) = \det(L_1) \neq 0$ *then*

$$\widetilde{\mathrm{Kh}}_\sigma(L) = H_*\left(\widetilde{\mathrm{Kh}}_\sigma(L_0)[-\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}_\sigma(L_1)\right).$$

Similarly, if $\det(L_1) = 0$ *and* $\det(L) = \det(L_0) \neq 0$ *then*

$$\widetilde{\mathrm{Kh}}_\sigma(L) = H_*\left(\widetilde{\mathrm{Kh}}_\sigma(L_0) \to \widetilde{\mathrm{Kh}}_\sigma(L_1)[\tfrac{1}{2}]\right).$$

Proof. The proof closely follows the argument in [31] establishing Proposition 2.6, and as such we will adopt the same notation. Throughout, $\sigma = \sigma(L)$, $\sigma_0 = \sigma(L_0)$ and $\sigma_1 = \sigma(L_1)$. There are 2 orientations to consider in each case, hence 4 cases to consider in total.

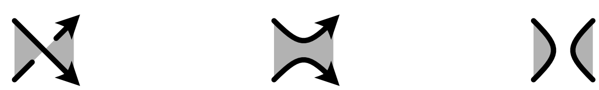


FIGURE 3. Colouring conventions for case 1: L, L_0 (the oriented resolution) and L_1 (the unoriented resolution) at the resolved positive crossing. For case 2 the white and black regions are exchanged to yield the dual colouring.

Case 1: Suppose the distinguished crossing is positive, with $\det(L_0) = 0$, and fix a checkerboard colouring of the diagram for L as in Figure 3 so that the distinguished crossing is of type II with incidence $\mu = +1$. Now writing G_1 for the Goeritz matrix of L_1, we have

$$G = \begin{pmatrix} a & v \\ v^T & G_1 \end{pmatrix} \quad \text{and} \quad G_0 = \begin{pmatrix} a-1 & v \\ v^T & G_1 \end{pmatrix}$$

where G and G_0 are the Goeritz matrices of L and L_0 respectively. As in [31], we assume without loss of generality that G_1 is diagonal (with diagonal entries $\alpha_1, \dots, \alpha_n$) and write the bilinear form associated to G as

$$\left(a - \sum_{i=1}^{n} \frac{v_i^2}{\alpha_i}\right) x_0^2 + \sum_{i=1}^{n} \alpha_i \left(x_i + \frac{v_i}{\alpha_i} x_0\right)^2.$$

Similarly, the bilinear form associated to G_0 may be written as

$$\left(a-1-\sum_{i=1}^{n}\frac{v_i^2}{\alpha_i}\right)x_0^2+\sum_{i=1}^{n}\alpha_i\left(x_i+\frac{v_i}{\alpha_i}x_0\right)^2$$

so that setting

$$\beta = a-\sum_{i=1}^{n}\frac{v_i^2}{\alpha_i}$$

we obtain

$$\det(G)=\beta\det(G_1) \text{ and } \det(G_0)=(\beta-1)\det(G_1).$$

Now since $0=\det(L_0)=|\det(G_0)|=|\beta-1|\det(L_1)$ and $\det(L_1)\neq 0$, we have that $\beta=+1$ and

$$\mathrm{signature}(G)=\mathrm{signature}(G_0)+1=\mathrm{signature}(G_1)+1.$$

Using the Gordon-Litherland formula for the signature we have that

$$\begin{aligned}\sigma &= \mathrm{signature}(G)-\mu\\ &= \mathrm{signature}(G_0)+1-(\mu_0+1)\\ &= \sigma_0\end{aligned}$$

where $\mu=\mu(L)$ and $\mu_0=\mu(L_0)$, while writing $\mu_1=\mu(L_1)$ gives

$$\begin{aligned}\sigma &= \mathrm{signature}(G)-\mu\\ &= \mathrm{signature}(G_1)+1-(\mu_1+c+1)\\ &= \sigma_1-c\end{aligned}$$

as in [31], noting that the incidence and type of a crossing determines its sign. Now since

$$\widetilde{\mathrm{Kh}}\,(L)\cong H_*\left(\widetilde{\mathrm{Kh}}\,(L_0)\,[-\tfrac{1}{2}]\to\widetilde{\mathrm{Kh}}\,(L_1)\,[-\tfrac{c}{2}]\right)$$

we have $-1=\sigma-\sigma_0-1$ and $-c=\sigma-\sigma_1$ so that

$$\widetilde{\mathrm{Kh}}\,(L)\,[-\tfrac{\sigma}{2}]\cong H_*\left(\widetilde{\mathrm{Kh}}\,(L_0)\,[-\tfrac{\sigma_0+1}{2}]\to\widetilde{\mathrm{Kh}}\,(L_1)\,[-\tfrac{\sigma_1}{2}]\right).$$

In terms of the σ-normalization,

$$\widetilde{\mathrm{Kh}}_\sigma(L)=H_*\left(\widetilde{\mathrm{Kh}}_\sigma(L_0)[-\tfrac{1}{2}]\to\widetilde{\mathrm{Kh}}_\sigma(L_1)\right)$$

as claimed.

Case 2: If once again we consider a positive distinguished crossing, but instead the resolution L_1 has $\det(L_1)=0$, then fix the dual colouring to that of Figure 3 so that the distinguished crossing is of type I with incidence $\mu=-1$. Now letting G, G_0 and G_1 be the Goeritz matrices for L, L_0 and L_1 respectively, we have that

$$G=\begin{pmatrix}a & v\\ v^T & G_0\end{pmatrix}\quad\text{and}\quad G_1=\begin{pmatrix}a+1 & v\\ v^T & G_0\end{pmatrix}$$

Diagonalizing yields

$$\det(G)=\beta\det(G_0) \text{ and } \det(G_1)=(\beta+1)\cdot\det(G_0)$$

so our hypothesis forces $\beta = -1$, resulting in

$$\operatorname{signature}(G) = \operatorname{signature}(G_0) - 1 = \operatorname{signature}(G_1) - 1.$$

Therefore,

$$\begin{aligned}\sigma &= \operatorname{signature}(G) - \mu \\ &= \operatorname{signature}(G_0) - 1 - \mu_0 \\ &= \sigma_0 - 1\end{aligned}$$

while

$$\begin{aligned}\sigma &= \operatorname{signature}(G) - \mu \\ &= \operatorname{signature}(G_1) - 1 - (\mu_1 + c) \\ &= \sigma_1 - c - 1\end{aligned}$$

so that $-1 = \sigma - \sigma_0$ and $c = \sigma - \sigma_1 + 1$. Thus

$$\widetilde{\mathrm{Kh}}\,(L) \cong H_*\left(\widetilde{\mathrm{Kh}}\,(L_0)\,[-\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}\,(L_1)\,[-\tfrac{c}{2}]\right)$$

yields

$$\widetilde{\mathrm{Kh}}\,(L)\,[-\tfrac{\sigma}{2}] \cong H_*\left(\widetilde{\mathrm{Kh}}\,(L_0)\,[-\tfrac{\sigma_0}{2}] \to \widetilde{\mathrm{Kh}}\,(L_1)\,[-\tfrac{\sigma_1-1}{2}]\right).$$

In terms of the σ-normalization,

$$\widetilde{\mathrm{Kh}}_\sigma(L) = H_*\left(\widetilde{\mathrm{Kh}}_\sigma(L_0) \to \widetilde{\mathrm{Kh}}_\sigma(L_1)[\tfrac{1}{2}]\right)$$

as claimed.

Case 3: Suppose the distinguished crossing is negative, with $\det(L_1) = 0$; the argument varies only slightly. This time, fixing the checkerboard colouring for the diagram of L so that the distinguished crossing is again of type II, the incidence is $\mu = -1$ (see Figure 4).

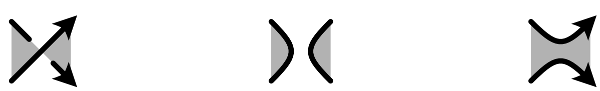


FIGURE 4. Colouring conventions for case 3: L, L_0 (the unoriented resolution) and L_1 (the oriented resolution) at the resolved negative crossing. For case 4 the white and black regions are exchanged to yield the dual colouring.

Following the conventions above, we have that

$$G = \begin{pmatrix} a & v \\ v^T & G_0 \end{pmatrix} \quad \text{and} \quad G_1 = \begin{pmatrix} a+1 & v \\ v^T & G_0 \end{pmatrix}$$

(notice that the resolutions exchange roles and have been renamed accordingly). Diagonalizing we obtain

$$\det(G) = \beta \det(G_0) \text{ and } \det(G_1) = (\beta + 1) \cdot \det(G_0)$$

so our hypothesis forces $\beta = -1$, resulting in

$$\text{signature}(G) = \text{signature}(G_0) - 1 = \text{signature}(G_1) - 1.$$

Now

$$\begin{aligned}\sigma &= \text{signature}(G) - \mu \\ &= \text{signature}(G_0) - 1 - (\mu_0 + c) \\ &= \sigma_0 - c - 1\end{aligned}$$

as in [31] while

$$\begin{aligned}\sigma &= \text{signature}(G) - \mu \\ &= \text{signature}(G_1) - 1 - (\mu_1 - 1) \\ &= \sigma_1.\end{aligned}$$

Finally, since

$$\widetilde{\mathrm{Kh}}\,(L) \cong H_*\left(\widetilde{\mathrm{Kh}}\,(L_0)\,[-\tfrac{c+1}{2}] \to \widetilde{\mathrm{Kh}}\,(L_1)\,[\tfrac{1}{2}]\right)$$

we conclude that

$$\widetilde{\mathrm{Kh}}\,(L)\,[-\tfrac{\sigma}{2}] \cong H_*\left(\widetilde{\mathrm{Kh}}\,(L_0)\,[-\tfrac{\sigma_0}{2}] \to \widetilde{\mathrm{Kh}}\,(L_1)\,[-\tfrac{\sigma_1-1}{2}]\right).$$

In terms of the σ-normalization,

$$\widetilde{\mathrm{Kh}}_\sigma(L) = H_*\left(\widetilde{\mathrm{Kh}}_\sigma(L_0) \to \widetilde{\mathrm{Kh}}_\sigma(L_1)[\tfrac{1}{2}]\right)$$

as claimed.

Case 4: With distinguished negative crossing but $\det(L_0) = 0$, we use the dual colouring to that of Figure 4, so that the distinguished crossing is of type I with incidence $\mu = +1$, and proceed as before. In this case we have

$$G = \begin{pmatrix} a & v \\ v^T & G_1 \end{pmatrix} \quad \text{and} \quad G_0 = \begin{pmatrix} a-1 & v \\ v^T & G_1 \end{pmatrix}$$

Diagonalizing yields

$$\det(G) = \beta\det(G_1) \text{ and } \det(G_0) = (\beta - 1)\cdot\det(G_1)$$

so our hypothesis forces $\beta = +1$, resulting in

$$\text{signature}(G) = \text{signature}(G_0) + 1 = \text{signature}(G_1) + 1.$$

Therefore,

$$\begin{aligned}\sigma &= \text{signature}(G) - \mu \\ &= \text{signature}(G_0) + 1 - (\mu_0 + c + 1) \\ &= \sigma_0 - c\end{aligned}$$

while

$$\begin{aligned}\sigma &= \text{signature}(G) - \mu \\ &= \text{signature}(G_1) + 1 - \mu_1 \\ &= \sigma_1 + 1\end{aligned}$$

so that

$$\widetilde{\mathrm{Kh}}\,(L) \cong H_*\left(\widetilde{\mathrm{Kh}}\,(L_0)\,[-\tfrac{c+1}{2}] \to \widetilde{\mathrm{Kh}}\,(L_1)\,[\tfrac{1}{2}]\right)$$

yields

$$\widetilde{\mathrm{Kh}}\,(L)\,[-\tfrac{\sigma}{2}] \cong H_*\left(\widetilde{\mathrm{Kh}}\,(L_0)\,[-\tfrac{\sigma_0+1}{2}] \to \widetilde{\mathrm{Kh}}\,(L_1)\,[-\tfrac{\sigma_1}{2}]\right)$$

In terms of the σ-normalization,

$$\widetilde{\mathrm{Kh}}_\sigma(L) = H_*\left(\widetilde{\mathrm{Kh}}_\sigma(L_0)[-\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}_\sigma(L_1)\right)$$

as claimed. □

3. Fillings, involutions and tangles

We review the basic notions of Dehn surgery that will be required. The material of this section is for the most part standard, see for example Boyer [8] and Rolfsen [47].

Let M be a compact, connected, orientable 3-manifold with torus boundary. A slope in ∂M is a class $\alpha \in H_1(\partial M;\mathbb{Z})/\pm 1$, that is, the isotopy class of an essential, simple, closed curve in ∂M. Since $H_1(\partial M;\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}$, the slopes in ∂M may be parameterized by reduced rational numbers $\{\frac{p}{q}\} \in \mathbb{Q}\cup\{\frac{1}{0}\}$ once a basis (α,β) for $H_1(\partial M;\mathbb{Z})$ has been fixed. That is, any slope may be written in the form $p\alpha + q\beta$ for relatively prime integers p and q, so that the slope α is represented by $\frac{1}{0}$. There is some redundancy in this description that may be taken care of by fixing the convention $q \geq 0$, say. Notice that, as a basis for $H_1(\partial M;\mathbb{Z})$, we have that α and β intersect geometrically in a single point. More generally, it will be useful to measure the distance between any two slopes by their geometric intersection number, denoted

$$\Delta(\alpha,\beta) = |\alpha\cdot\beta|\,,$$

for any $\alpha,\beta \in H_1(M;\mathbb{Z})/\pm 1$.

For any slope α, denote by $M(\alpha)$ the result of Dehn filling along α. For example, given a knot $K \hookrightarrow S^3$, denote the complement $M = S^3 \smallsetminus \nu(K)$ where $\nu(K)$ is an open tubular neighbourhood of the knot $K \hookrightarrow S^3$. In this setting there is a preferred basis for surgery provided by the knot meridian μ, and the longitude of the knot λ resulting from the fact that K bounds an oriented surface (a Seifert surface) in S^3. We may choose orientations on μ and λ so that $\mu\cdot\lambda = 1$, and this convention will be assumed throughout.

Now if α is a slope in the boundary of the knot complement, we may write $p\mu + q\lambda$ for $q \geq 0$. This gives rise to the notation $M(\alpha) = S^3_{p/q}(K)$ for Dehn filling, referred to as surgery on K. This fixes the convention $\frac{1}{0} = \infty$ for the trivial surgery $S^3_{1/0}(K) \cong S^3$. By nature of this construction, we have that

$$\left|H_1(S^3_{p/q}(K);\mathbb{Z})\right| = \left|H_1(M(\alpha);\mathbb{Z})\right| = \Delta(\alpha,\lambda)$$

(see, more generally, Lemma 3.2 below).

3.1. **The rational longitude.** For M as above, suppose that $H_1(M;\mathbb{Q}) \cong \mathbb{Q}$. Such manifolds M will be referred to as *knot manifolds*. Unless stated otherwise, we will generally make the additional assumption that a knot manifold M is irreducible. However, this is not an essential hypothesis in the following discussion, or in the proof of Lemma 3.2 below.

Let $i\colon \partial M \hookrightarrow M$ be the inclusion map, inducing a homomorphism
$$i_*\colon H_1(\partial M;\mathbb{Q}) \to H_1(M;\mathbb{Q}).$$
Omitting the coefficients for brevity, consider the long exact sequence
$$\cdots \longrightarrow H_2(M) \longrightarrow H_2(M,\partial M) \longrightarrow H_1(\partial M) \xrightarrow{i_*} H_1(M) \longrightarrow H_1(M,\partial M) \longrightarrow \cdots$$
Since ∂M is connected, the inclusion i induces an isomorphism $H_0(\partial M) \cong H_0(M)$ so that
$$0 \longrightarrow H_2(M) \longrightarrow H_2(M,\partial M) \longrightarrow H_1(\partial M) \xrightarrow{i_*} H_1(M) \longrightarrow H_1(M,\partial M) \longrightarrow 0$$
Since we are working over a field, by duality we have
$$H_2(M) \cong H^1(M,\partial M) \cong H_1(M,\partial M)$$
and
$$H_2(M,\partial M) \cong H^1(M) \cong H_1(M)$$
hence
$$0 \longrightarrow H_1(M,\partial M) \longrightarrow H_1(M) \longrightarrow H_1(\partial M) \xrightarrow{i_*} H_1(M) \longrightarrow H_1(M,\partial M) \longrightarrow 0$$
Now we observe that $\mathrm{rk}(i_*) = 1$, and this implies that $i_*\colon H_1(\partial M;\mathbb{Z}) \to H_1(M;\mathbb{Z})$ carries a free summand of $H_1(\partial M;\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}$ injectively to $H_1(M;\mathbb{Z}) \cong \mathbb{Z}\oplus H$ (for some finite abelian group H). Moreover, $\ker(i_*)$ must be generated generated by $k\lambda_M$, for some primitive class $\lambda_M \in H_1(\partial M;\mathbb{Z})$, and non-zero integer k.

Note that this class is uniquely defined, up to sign, and hence determines a well-defined slope in ∂M. This gives a preferred slope in ∂M for any knot manifold, and in turn motivates the following definition.

Definition 3.1. *For any knot manifold M, the rational longitude λ_M is the unique slope with the property that $i_*(\lambda_M)$ is finite order in $H_1(M;\mathbb{Z})$.*

More geometrically, the rational longitude λ_M is characterized among all slopes by the property that a finite number of like-oriented parallel copies of λ_M bounds an essential surface in M.

As with the preferred longitude for a knot in S^3, the rational longitude controls the first homology of the manifold obtained by Dehn filling.

Lemma 3.2. *For every knot manifold M there is a constant c_M (depending only on M) such that*
$$|H_1(M(\alpha);\mathbb{Z})| = c_M\Delta(\alpha,\lambda_M).$$

Proof. Orient λ_M and fix a curve μ dual to λ_M so that $\mu\cdot\lambda_M = 1$. This provides a choice of basis (μ,λ_M) for the group $H_1(\partial M;\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}$. Under the homomorphism induced by inclusion we

have $i_*(\mu) = (\ell, u)$ and $i_*(\lambda_M) = (0, h)$ as elements of $H_1(M;\mathbb{Z}) \cong \mathbb{Z} \oplus H$. Note that for any other choice of class μ' such that $\mu' \cdot \lambda_M = 1$ we have $\mu' = \mu + n\lambda_M$ so that $i_*(\mu') = (\ell, u + nh)$.

We claim that $|\ell| = \operatorname{ord}_H i_*(\lambda_M)$.

Let ζ generate a free summand of $H_1(M;\mathbb{Z})$ so that (the free part of) the image of μ is $\ell\zeta$ where $i_*(\mu) = (\ell, u) \in \mathbb{Z} \oplus H$, and let η generate the free part of $H_2(M, \partial M;\mathbb{Z})$. Then $\eta \cdot \zeta = \pm 1$ under the intersection pairing $H_2(M, \partial M;\mathbb{Z}) \otimes H_1(M;\mathbb{Z}) \to \mathbb{Z}$.

Now suppose $k = \operatorname{ord}_H i_*(\lambda_M)$. The long exact sequence in homology gives

$$\begin{tikzcd} \cdots \arrow[r] & H_2(M;\mathbb{Z}) \arrow[r] & H_2(M,\partial M;\mathbb{Z}) \arrow[r, "\partial"] & H_1(\partial M;\mathbb{Z}) \arrow[r, "i_*"] & H_1(M;\mathbb{Z}) \arrow[r] & \cdots \\ & & \theta \arrow[r, mapsto] & k\lambda_M \arrow[r, mapsto] & 0 & \end{tikzcd}$$

so there is a class $\theta \in H_2(M, \partial M;\mathbb{Z})$ with image $k\lambda_M$. Now we have already observed in defining λ_M that $\operatorname{rk}(i_*) = 1$, and hence $\theta = a\eta$ for some integer $a \neq 0$. Therefore $k\lambda_M = a\partial\eta$, hence $\partial\eta = \frac{k}{a}\lambda_M$. But since $i_*(\frac{k}{a}\lambda_M) = 0$, it must be that $|\frac{k}{a}| = |k|$ so that $|a| = 1$. As a result, $\theta = \pm\eta$. In particular, up to a choice of sign $\partial\eta = k\lambda_M$ as an element of $H_1(\partial M;\mathbb{Z})$. Now

$$|k| = |\mu \cdot k\lambda_M| = |\mu \cdot \partial\eta| = |\ell\zeta \cdot \eta| = |\ell|$$

as claimed.

For a given slope α write $\alpha = a\mu + b\lambda_M$ so that $i_*(\alpha) = (a\ell, au + bh)$. Then

$$H_1(M(\alpha);\mathbb{Z}) \cong H_1(M;\mathbb{Z})/(a\ell, au + bh)$$

has presentation matrix of the form

$$\begin{pmatrix} a\ell & 0 \\ au + bh & Ir \end{pmatrix}$$

where $r = (r_1, \ldots, r_n)$ specifies the finite abelian group $H = \mathbb{Z}/r_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/r_n\mathbb{Z}$. Therefore $|H_1(M(\alpha);\mathbb{Z})| = a\ell r_i \cdots r_n$. Setting

$$c_M = \ell r_1 \cdots r_n = (\operatorname{ord}_H i_*(\lambda_M))|H|$$

and noting that $a = \Delta(\alpha, \lambda_M)$ proves the lemma. □

3.2. **Strong inversions and associated quotient tangles.** A knot manifold is called strongly invertible if there is an involution $f\colon M \to M$ with 1-dimensional fixed point set intersecting the boundary torus transversely in exactly 4 points. More precisely,

$$\operatorname{Fix}(f) \cong I \amalg I \amalg \underbrace{S^1 \amalg \cdots \amalg S^1}_{k}$$

where $k \geq 0$. A knot is called strongly invertible if its complement is strongly invertible.

Definition 3.3. *Given an irreducible knot manifold M with $H_1(M;\mathbb{Q}) = \mathbb{Q}$, suppose that there is a strong inversion $f \in \operatorname{End}(M)$ with the property that M/f is homeomorphic to a ball. Such M will be called a simple, strongly invertible knot manifold.*

As a result, any simple strongly invertible knot manifold is naturally the two fold branched cover of a tangle $\mathbf{\Sigma}(B^3,\tau)$, where τ is a pair of properly embedded arcs in B^3 given by the image of $\mathrm{Fix}(f)$ in the quotient.

Definition 3.4. *To any simple, strongly invertible knot manifold M, let the associated quotient tangle $T=(B^3,\tau)$ is obtained by taking $\tau=\mathrm{image}(\mathrm{Fix}(f))$.*

In this setting, equivalence of tangles is taken up to homeomorphism of the pair (in the sense of Lickorish [30]), and need not fix the boundary in general. Note that the solid torus is a simple, strongly invertible knot manifold. Indeed, according to Lickorish, a tangle is rational if and only if the two-fold branched cover is a solid torus [30]. As a result, a strong inversion may always be extended across a surgery torus (moreover, the resulting branch set may be recovered, see Proposition 3.7). This fact is generally attributed to Montesinos [33]. In particular, we have:

Proposition 3.5. *When $K\hookrightarrow S^3$ is strongly invertible, the complement $M\smallsetminus\nu(K)$ is simple.*

Proof. Extending f to S^3, across the surgery torus of the trivial surgery, gives the standard involution on S^3 by definition of strong invertibility. The quotient of this involution is S^3, decomposed along a sphere obtained by the quotient of the torus ∂M. Since S^3 decomposes into a pair of 3-balls for any embedding $S^2\hookrightarrow S^3$, M/f must therefore be homeomorphic to B^3. □

For example, the trefoil is a strongly invertible knot, and the associated quotient tangle is constructed in Figure 5. While complements of strongly invertible knots in S^3 provide the primary source of examples of simple, strongly invertible knot manifolds, we remark that the latter is certainly a much larger class. For example, the exterior of a generalized torus knot – those manifolds Seifert fibered over the disk with two cone points – always provides such a manifold.

Proposition 3.6 (Montesinos [34])**.** *Let Y be a Seifert fibre space with base orbifold $S^2(p,q,r)$. Then $Y\cong M(\alpha)$ where M is a simple strongly invertible knot manifold and M has Seifert fibre structure with base orbifold $D^2(p,q)$.*

Proof. Let M be a knot manifold endowed with a Seifert fibre structure and suppose that the base orbifold is $D^2(p,q)$, the disk with two cone points. We may assume that $D^2=\{z\in\mathbb{C}:|z|\le 1\}$, and that the cone points p,q lie on the real axis in the interior of D^2. Note that such a Seifert fibre space is a union of solid tori along an essential annulus that corresponds to the lift of the imaginary axis in the interior of D^2. As we have noted previously, the solid torus admits a strong inversion, and such a strong inversion fixes the singular fibre of any Seifert fibre structure on the solid torus. In particular, the solid torus as a Seifert fibre space has base orbifold D^2 with a single cone point, and the strong inversion corresponds to a reflection in the real axis. Now the reflection $\rho(z)=\bar{z}$ in the real axis (fixing the cone points p,q) lifts to a strong inversion on M, and ρ fixes the singular fibres.

Choose a regular fibre $\varphi\subset\partial M$. By a theorem of Heil, the Dehn filling $M(\varphi)$ must be a connect sum of lens spaces [19]. Further, extending the strong inversion across the surgery torus gives a strong inversion on $M(\varphi)$, the quotient of which is S^3 (with branch set a connect sum of 2-bridge links [20]). As a result, $M/f\cong B^3$ as in the proof of Proposition 3.5.

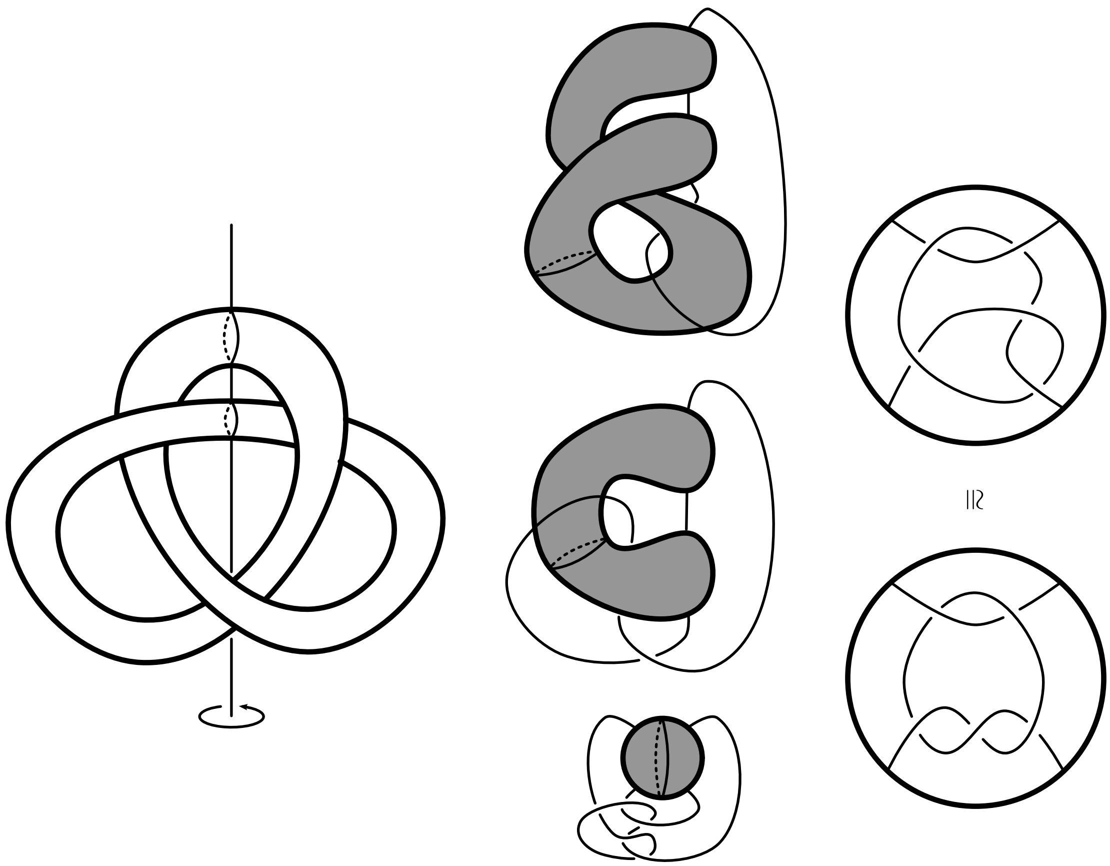


Figure 5. The trefoil with its strong inversion (left), an isotopy of a fundamental domain for the involution (centre), and two homeomorphic views of the tangle associated to the quotient (right). Notice that both representatives of the tangle have the property that $\tau(\frac{1}{0})$ is the trivial knot, giving a branch set for the trivial surgery. With a little more care, one may keep track of the image of the preferred longitude in the quotient; see Bleiler [5], for example.

Now suppose that Y is Seifert fibered, with base orbifold $S^2(p,q,r)$. Removing a tubular neighbourhood of a singular fibre yields a knot manifold M that is Seifert fibered with base orbifold $D^2(p,q)$. Such an M must be simple and strongly invertible. □

Two examples that will be particularly useful in the sequel are the twisted I-bundle over the Klein bottle (this is the unique manifold admitting a $D^2(2,2)$ Seifert structure) and the complement of the trefoil knot in S^3 (admitting a $D^2(2,3)$ Seifert structure; this is unique up to mirrors).

3.3. **Tangles and Dehn filling.** For a given simple strongly invertible knot manifold M, any representative of the associated quotient tangle T (in particular, having fixed choice of 4 points $\partial\tau \hookrightarrow \partial B^3$) has a pair of distinguished arcs $(\gamma_{\frac{1}{0}}, \gamma_0)$ in the boundary of the tangle, as illustrated

in Figure 6, that meet in a single point. The hemisphere containing each arc lifts to an annulus in $\partial M = \mathbf{\Sigma}(\partial B^3, \partial\tau)$, so that the pair $(\gamma_{\frac{1}{0}}, \gamma_0)$ lifts to a (unoriented) basis for $H_1(\partial M; \mathbb{Z})$. By fixing an orientation so that $\widetilde{\gamma}_{\frac{1}{0}} \cdot \widetilde{\gamma}_0 = 1$, we obtain a basis for Dehn fillings of M.

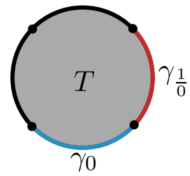


FIGURE 6. The arcs $\gamma_{\frac{1}{0}}$ (red) and γ_0 (blue) in the boundary of T.

We will make use of an action 3-strand braid group $B_3 = \langle \sigma_1, \sigma_2 | \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$ on the space of tangles, $\mathcal{T}$. Braids in this setting are depicted horizontally, read from left to right, with standard generators

$\sigma_1 =$  $\sigma_2 =$ 

For a given braid $\beta \in B_3$ the action

$$\begin{aligned} \mathcal{T} \times B_3 &\to \mathcal{T} \\ (T, \beta) &\mapsto T^\beta \end{aligned}$$

is defined by taking T^β as the tangle depicted in Figure 7.

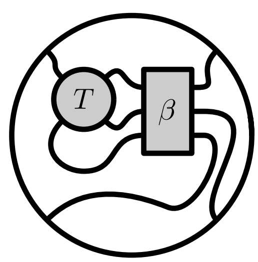


FIGURE 7. The tangle T^β.

It is straightforward to verify that this is a well defined action on tangles. Notice that this specifies a homeomorphism of the given tangle, and as such this action is trivial when considering tangles up to homeomorphism (though the choice of diagram for a fixed tangle may be altered dramatically). In fact, this may be viewed as a change of framing.

Let $\frac{p}{q} = [a_1, \ldots, a_r]$ be the continued fraction expansion with $a_1 \geq 0$ and $a_i > 0$ for $i > 1$ when $\frac{p}{q} \geq 0$ (when $\frac{p}{q} \leq 0$, $a_1 \leq 0$ and $a_i < 0$ for $i > 1$). To $\frac{p}{q}$ we associate the braid

$$\beta = \begin{cases} \sigma_2^{a_1}\sigma_1^{-a_2}\cdots\sigma_1^{-a_r} & r \text{ even} \\ \sigma_2^{a_1}\sigma_1^{-a_2}\cdots\sigma_2^{a_r} & r \text{ odd} \end{cases}$$

Define the *odd* and *even* closures of T (also referred to as the *numerator* and *denominator* closures, respectively), as in Figure 8. Now observing that $0 = [0]$, and fixing the convention $\frac{1}{0} = [\ ]$ (with *length* $r = 0$), denote $\tau(\frac{p}{q})$ the link obtained by the even or odd closure of T^β depending on whether r is even or odd (a particular example is shown in Figure 9).

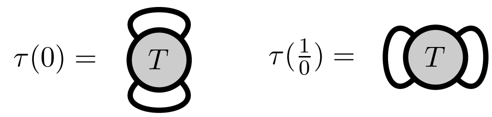


FIGURE 8. The odd-closure $\tau(0)$ and the even-closure $\tau(\frac{1}{0})$ of the tangle T.

Now the strong inversion on M extends to an involution on a Dehn filling of M, giving rise to a two-fold branched cover of S^3, branched over a link that we may now make explicit.

Proposition 3.7. *Let M be a simple strongly invertible knot manifold. For a given slope $\alpha = p\widetilde{\gamma}_{\frac{1}{0}} + q\widetilde{\gamma}_0$ we have that $\mathbf{\Sigma}(S^3, \tau(\frac{p}{q})) \cong M(\alpha)$.*

Sketch of proof. First observe that $\mathbf{\Sigma}(S^3, \tau(0)) \cong M(\widetilde{\gamma}_0)$ and $\mathbf{\Sigma}(S^3, \tau(\frac{1}{0})) \cong M(\widetilde{\gamma}_{\frac{1}{0}})$.

Now consider the action of σ_2. We claim that this half twist (viewed as an action on the disk with 2 marked points) lifts to a Dehn twist along the curve $\widetilde{\gamma}_{\frac{1}{0}}$. Indeed, the two fold branched cover of this disk is an essential annulus in ∂M (c.f [47, Chapter 10]). In terms of the basis $(\widetilde{\gamma}_{\frac{1}{0}}, \widetilde{\gamma}_0)$, this Dehn twist may be written $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Similarly, the action of σ_1^{-1} lifts to a Dehn twist about $\widetilde{\gamma}_0$; this takes the form $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. In particular, we have that $\mathbf{\Sigma}(S^3, \tau(n)) \cong M(n\widetilde{\gamma}_{\frac{1}{0}})$ and $\mathbf{\Sigma}(S^3, \tau(\frac{1}{n})) \cong M(n\widetilde{\gamma}_0)$.

In general, for $\frac{p}{q} = [a_1, \ldots, a_r]$, the action of the associated braid may be written (in the case r is even) as

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{a_r} \cdots \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{a_2} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{a_1}$$

(the case r odd differs only in the first matrix of this product). We leave it to the reader to check that the first column of the resulting matrix is $\begin{pmatrix} q \\ p \end{pmatrix}$ so that we have specified the filling slope $\alpha = p\widetilde{\gamma}_{\frac{1}{0}} + q\widetilde{\gamma}_0$ as desired. Details may be found in Rolfsen [47, Chapter 10], see also Montesinos [33]. $\square$

Corollary 3.8. *Given a basis (α, β) for surgery in ∂M there is a choice of representative for T so that $(\gamma_{\frac{1}{0}}, \gamma_0)$ lifts to (α, β).*

Proof. For any choice of representative of T, write $\alpha = p\widetilde{\gamma}'_{\frac{1}{0}} + q\widetilde{\gamma}'_0$. In in terms of this representative then, $M(\alpha) = \mathbf{\Sigma}(S^3, \tau(\frac{p}{q}))$. However, by removing the arcs forming the closure as in Figure 8,

the resulting tangle may be viewed as a reframing if T, and yields a representative compatible with α. By twisting along α (i.e. by half twists in the quotient), this representative may be made compatible with (α, β) since $\alpha \cdot \beta = 1$. $\square$

As a result, for any choice of basis (α, β) for Dehn surgery on a simple strongly invertible knot manifold, a *compatible* representative for the associated quotient tangle exists so that $\alpha = \widetilde{\gamma}_{\frac{1}{0}}$ and $\beta = \widetilde{\gamma}_0$. Notice that, as a result of Lemma 3.2, we have that

$$\det(\tau(\tfrac{p}{q})) = c_M \Delta(p\widetilde{\gamma}_{\frac{1}{0}} + q\widetilde{\gamma}_0, \lambda_M) = c_M \Delta(p\alpha + q\beta, \lambda_M)$$

once a basis for Dehn surgery, and compatible associated quotient tangle have been fixed. In particular, given a strongly invertible knot in S^3 there is always a choice of associated quotient tangle for which

$$S^3_{p/q}(K) = \mathbf{\Sigma}(S^3, \tau(\tfrac{p}{q})).$$

Such a representative will be referred to as the preferred representative for the associated quotient tangle.

Note that $S^3_{p/q}(K) \cong -S^3_{-p/q}(K^\star)$ where $-Y$ denotes the manifold Y with orientation reversed, and $K^\star$ denotes the mirror image of K. More generally,

$$\mathbf{\Sigma}(S^3, \tau(\tfrac{p}{q})) \cong -\mathbf{\Sigma}(S^3, \tau(\tfrac{p}{q})^\star) \cong -\mathbf{\Sigma}(S^3, \tau^\star(-\tfrac{p}{q})),$$

and as a consequence we will only need to consider non-negative surgery coefficients and continued fractions (up to taking mirrors).

3.4. **Some properties of continued fractions.** There are three fundamental properties for continued fractions relating to Dehn filling that will be essential for the inductive arguments that follow. Since it will always be possible to restrict to non-negative surgery coefficients by passing to the mirror image, we will state these properties for non-negative continued fractions only.

Therefore, assume that $\frac{p}{q} = [a_1, \ldots, a_r]$ is non-negative, with $a_1 \geq 0$ and $a_i > 0$ for all $i > 1$

Property 3.9. $\lfloor \frac{p}{q} \rfloor = a_1$ *and* $\lceil \frac{p}{q} \rceil = a_1 + 1$.

Proof. It is immediate from the definition of $\frac{p}{q}$ as a continued fraction that $a_1 \leq \frac{p}{q} < a_1 + 1$ for $\frac{p}{q} = [a_1, \ldots, a_r]$. $\square$

Property 3.10. $[a_1, \ldots, a_r, 1] = [a_1, \ldots, a_r + 1]$.

Proof. This is immediate from the partial evaluation of the continued fraction:

$$[a_1, \ldots, a_r, 1] = [a_1, \ldots, a_r + \tfrac{1}{1}] = [a_1, \ldots, a_r + 1]$$

$\square$

It is important to note that this equality of continued fractions manifests itself as isotopic links when forming $\tau(\frac{p}{q})$, for any tangle. This results from the fact that the even- and odd-closures replace one another, as is illustrated in a particular case in Figure 9.

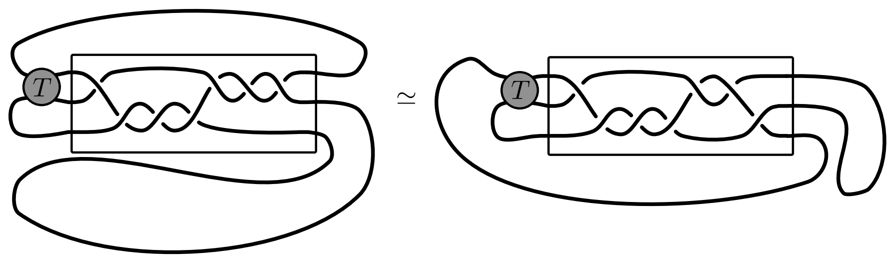


FIGURE 9. The link $\tau(\frac{13}{10})$ obtained from the odd-closure with the fraction $[1,3,3]$ (left), is isotopic to the link obtained from the even-closure with the fraction $[1,3,2+1]=[1,3,2,1]$ (right).

Finally, we turn to the behaviour of $\tau(\frac{p}{q})$ under resolutions.

Definition 3.11. *The terminal crossing of $\tau(\frac{p}{q})$ is the last crossing added by the action of $\beta \in B_3$ specified by the continued fraction. That is, the terminal crossing corresponds to the last generator in the braid word $\beta = \sigma_2^{a_1}\cdots\sigma_\epsilon^{a_r}$ (where σ_ϵ is either σ_2 or σ_1^{-1}, depending on the parity of r).*

Our convention will be that the terminal crossing of $\tau(\frac{p}{q})$ is resolved to obtain the 0-resolution $\tau(\frac{p_0}{q_0})$ and the 1-resolution $\tau(\frac{p_1}{q_1})$. Notice that the 0-resolution is given by one of $[a_1,\ldots,a_{r-1}]$ or $[a_1,\ldots,a_{r-1},a_r-1]$ depending on the parity of r, and the 1-resolution is given by the other. By Property 3.10 we may assume without loss of generality that $a_r>1$ when $\frac{p}{q}=[a_1,\ldots,a_r]$.

Property 3.12. $\frac{p}{q}=\frac{p_0+p_1}{q_0+q_1}$ *where* $\frac{p_0}{q_0}$ *and* $\frac{p_1}{q_1}$ *are the continued fractions associated to the 0- and 1-resolution, respectively.*

Proof. Recall that a continued fraction may be recursively defined by convergents $\frac{h_n}{k_n}$ where $h_{-1}=0, h_0=1$ and $h_n=a_nh_{n-1}+h_{n-2}$ for $n>1$, and $k_{-1}=1, k_0=0$ and $k_n=a_nk_{n-1}+k_{n-2}$ for $n>1$.

Now write $\frac{h_{r-1}}{k_{r-1}}=\frac{p_0}{q_0}$ and $\frac{h_r}{k_r}=\frac{p_1}{q_1}$, then $\frac{p_0+p_1}{q_0+q_1}=\frac{h_r+h_{r-1}}{k_r+k_{r-1}}=[a_1,\ldots,a_r-1,1]$, so that applying Property 3.10 we have $\frac{p_0+p_1}{q_0+q_1}=[a_1,\ldots,a_r]=\frac{p}{q}$ as claimed. □

A particular example of Property 3.12 is illustrated in Figure 10. When $\frac{p}{q}=[a_1,\ldots,a_r]$ we will use the notation $\tau(\frac{p}{q})=\tau[a_1,\ldots,a_r]$ for the closure where convenient.

4. Branch sets and width bounds

This section has two principle aims. First, we investigate the relationship between Dehn fillings of simple, strongly invertible knot manifolds and quasi-alternating links. The remaining sections are devoted to establishing bounds on the homological width of branch sets associated to Dehn surgery on strongly invertible knots in S^3.

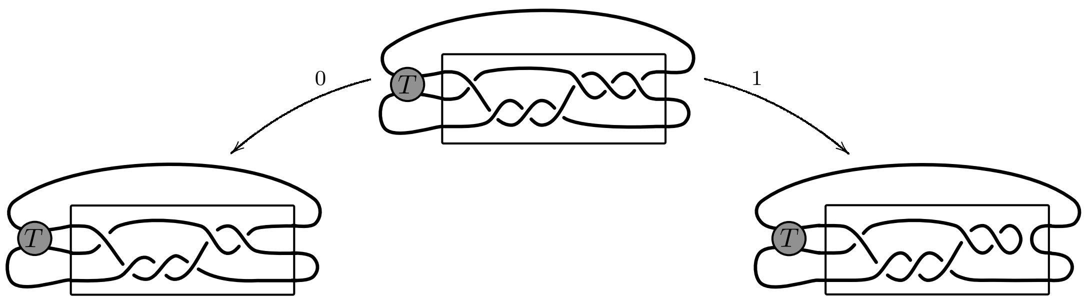


FIGURE 10. Resolving the terminal crossing of $\tau(\frac{13}{10}) = \tau[1,3,3]$ gives 0-resolution with $\frac{p_0}{q_0} = [1,3,2] = \frac{9}{7}$ and 1-resolution with $\frac{p_1}{q_1} = [1,3] = \frac{4}{3}$.

4.1. **Quasi-alternating links.** Given a strongly invertible knot $K \hookrightarrow S^3$, with fixed strong inversion, let $T = (B^3, \tau)$ be the associated quotient tangle. Choosing the preferred representative for T, we have that $\det(\tau(n)) = |H_1(\mathbf{\Sigma}(S^3, \tau(n)); \mathbb{Z})| = |H_1(S^3_n(K); \mathbb{Z})| = n$ for $n \geq 0$. Moreover, resolving the terminal crossing of $\tau(n+1)$ we have $\det(\tau(n+1)) = \det(\tau(n)) + \det(\tau(\frac{1}{0}))$. This leads naturally to the notion of a quasi-alternating link.

Definition 4.1. *The set of quasi-alternating links $\mathcal{Q}$ is the smallest set of links containing the trivial knot, and closed under the following relation: if L admits a projection with distinguished crossing $L(\times)$ so that*

$$\det(L(\times)) = \det(L(\asymp)) + \det(L(\,)\,(\,))$$

for which $L(\asymp), L(\,)\,(\,) \in \mathcal{Q}$, then $L = L(\times) \in \mathcal{Q}$ as well.

Ozsváth and Szabó show that non-split, alternating links are quasi-alternating, and that $\mathbf{\Sigma}(S^3, L)$ is an L-space whenever L is quasi-alternating [41]. Recall that an L-space is a rational homology sphere Y for which $\operatorname{rk}\widehat{\mathrm{HF}}(Y) = |H_1(Y; \mathbb{Z})|$ where $\widehat{\mathrm{HF}}$ denotes the Heegaard-Floer homology of Y (with $\mathbb{F}$ coefficients). Manolescu and Ozsváth have shown that quasi-alternating links are homologically thin [31], generalizing Lee's result that non-split alternating links are thin [29]. More generally:

Proposition 4.2. *The two-fold branched cover of a thin link is always an L-space.*

Proof. When L is a thin link, $\operatorname{rk}\widetilde{\mathrm{Kh}}(L) = \det(L)$. However, as a result of the spectral sequence relating Khovanov homology and Heegaard-Floer homology for two-fold branched covers, $\det(L) \leq \operatorname{rk}\widehat{\mathrm{HF}}(\mathbf{\Sigma}(S^3, L)) \leq \operatorname{rk}\widetilde{\mathrm{Kh}}(L)$ [41, Corollary 1.2]. Since $\det(L) = |H_1(\mathbf{\Sigma}(S^3, L); \mathbb{Z})|$, the result follows. $\square$

Note however that the converse is false: the Poincaré homology sphere arises as the two-fold branched cover of 10_{124} (shown in Figure 1). This manifold is an L-space with thick branch set.

Let M be a simple, strongly invertible, knot manifold. Suppose α and β are a pair of slopes in ∂M with $\alpha \cdot \beta = +1$. Fix a compatible representative for the associated quotient tangle $T = (B^3, \tau)$ with the property that $M(\alpha) = \mathbf{\Sigma}(S^3, \tau(\frac{1}{0}))$ and $M(\beta) = \mathbf{\Sigma}(S^3, \tau(0))$.

Proposition 4.3. *If $\tau(\frac{1}{0})$ and $\tau(0)$ are quasi-alternating, and $\alpha \cdot \lambda_M, \beta \cdot \lambda_M > 0$, then $\tau(1)$ is quasi-alternating as well.*

Remark 4.4. *Note that the quasi-alternating hypothesis ensures that $\det(\tau(\frac{1}{0}))$ and $\det(\tau(0))$ must be non-zero, hence neither α nor β coincides with the rational longitude.*

Proof of Proposition 4.3. We need to calculate $\det(\tau(1))$. To this end, by applying Lemma 3.2 we have that

$$\begin{aligned}
\det(\tau(1)) &= |H_1(M(\alpha+\beta);\mathbb{Z})| \\
&= c_M \Delta(\alpha+\beta, \lambda_M) \\
&= c_M |\alpha + \beta \cdot \lambda_M| \\
&= c_M |\alpha \cdot \lambda_M + \beta \cdot \lambda_M| \\
&= c_M |\alpha \cdot \lambda_M| + c_M |\beta \cdot \lambda_M| \\
&= c_M \Delta(\alpha, \lambda_M) + c_M \Delta(\beta, \lambda_M) \\
&= |H_1(M(\alpha);\mathbb{Z})| + |H_1(M(\beta);\mathbb{Z})| \\
&= \det(\tau(\tfrac{1}{0})) + \det(\tau(0)),
\end{aligned}$$

which verifies that $\tau(1)$ is a quasi-alternating link, since both $\tau(\frac{1}{0})$ and $\tau(0)$ are quasi-alternating by hypothesis. □

Remark 4.5. *The condition on intersection with λ_M may be relaxed at the expense of taking mirrors. For any $M(\alpha)$ and $M(\beta)$ with quasi-alternating branch sets $\tau(\frac{1}{0})$ and $\tau(0)$ respectively, we can ensure positive intersection with λ_M at the expense of $\alpha \cdot \beta = \pm 1$. In the case that $\alpha \cdot \beta = -1$, the same argument works by passing to mirrors. Any quasi-alternating link has quasi-alternating mirror image and as a result if $\tau(\frac{1}{0})$ and $\tau(0)$ are quasi-alternating than one of $\tau(-1)$ or $\tau(1)$ is quasi-alternating.*

Definition 4.6. *A triad of links $(\tau(\frac{1}{0}), \tau(0), \tau(1))$ corresponds to a triple of slopes $(\alpha, \beta, \alpha+\beta)$ in the boundary of $M = \mathbf{\Sigma}(B^3, \tau)$ where $\alpha \cdot \beta = 1$, $\alpha \cdot \lambda_M > 0$, and $\beta \cdot \lambda_M > 0$.*

The requirement that α and β intersect positively with λ_M is stronger than necessary, since it is attainable up to taking mirrors. However, with this assumption we have:

Theorem 4.7. *A triad of links, for which $\tau(\frac{1}{0})$ and $\tau(0)$ are quasi-alternating, gives rise to an infinite family of quasi-alternating links $\tau(\frac{p}{q})$, for $\frac{p}{q} \geq 0$.*

Proof. First observe that $\tau(n)$ is quasi-alternating for every $n \geq 0$. This is immediate by induction in n, since $\tau(0)$ is quasi-alternating and

$$\begin{aligned}
\det(\tau(n)) &= |H_1(M(n\alpha+\beta);\mathbb{Z})| \\
&= c_M\Delta(n\alpha+\beta,\lambda_M) \\
&= c_M|n\alpha+\beta\cdot\lambda_M| \\
&= c_M|n\alpha\cdot\lambda_M+\beta\cdot\lambda_M| \\
&= c_M|\alpha\cdot\lambda_M| + c_M|(n-1)\alpha+\beta\cdot\lambda_M| \\
&= c_M\Delta(\alpha,\lambda_M) + c_M\Delta((n-1)\alpha+\beta,\lambda_M) \\
&= |H_1(M(\alpha);\mathbb{Z})| + |H_1(M((n-1)\alpha+\beta);\mathbb{Z})| \\
&= \det(\tau(\tfrac{1}{0})) + \det(\tau(n-1)),
\end{aligned}$$

for $n > 0$ as in Proposition 4.3.

For $\tau(\frac{p}{q})$, we need a second induction in the length of the continued fraction $\frac{p}{q} = [a_1,\ldots,a_r]$. The base case $r = 1$ is the observation above that $\tau(n)$ is quasi-alternating, applying Property 3.9.

Suppose then that $\tau(\frac{p}{q})$ is quasi-alternating for all $\frac{p}{q} \geq 0$ that may be represented by a continued fraction of length $r-1$. By resolving the terminal crossing and applying Property 3.12 for $\frac{p}{q} = [a_1,\ldots,a_r]$,

$$\begin{aligned}
\det(\tau(\tfrac{p}{q})) &= |H_1(M(p\alpha+q\beta);\mathbb{Z})| \\
&= c_M\Delta(p\alpha+q\beta,\lambda_M) \\
&= c_M|p\alpha+q\beta\cdot\lambda_M| \\
&= c_M|(p_0+p_1)\alpha\cdot\lambda_M+(q_0+q_1)\beta\cdot\lambda_M| \\
&= c_M|(p_0\alpha+q_0\beta)\cdot\lambda_M| + c_M|(p_1\alpha+q_1\beta)\cdot\lambda_M| \\
&= c_M\Delta(p_0\alpha+q_0\beta,\lambda_M) + c_M\Delta(p_1\alpha+q_1\beta,\lambda_M) \\
&= |H_1(M(p_0\alpha+q_0\beta);\mathbb{Z})| + |H_1(M(p_1\alpha+q_1\beta);\mathbb{Z})| \\
&= \det(\tau(\tfrac{p_0}{q_0})) + \det(\tau(\tfrac{p_1}{q_1}))
\end{aligned}$$

where $\frac{p_0}{q_0}$ and $\frac{p_1}{q_1}$ are the continued fractions $[a_1,\ldots,a_{r-1}]$ and $[a_1,\ldots,a_r-1]$. This gives a continued fraction of length $r-1$ for which the corresponding link must be quasi-alternating by the induction hypothesis, and a continued fraction $[a_1,\ldots,a_r-1]$ with r^{th} entry reduced by one.

Since $[a_1,\ldots,a_{r-1},1] = [a_1,\ldots,a_{r-1}+1]$ by Property 3.10, repeating the above argument $a_r - 1$ times (i.e. a second induction in a_r as in the case $\tau(n)$) completes the induction. □

4.2. **Branch sets for L-spaces obtained from Berge knots.** Theorem 4.7 gives a tool with which build large classes of quasi-alternating links and study the overlap between certain classes of L-spaces. In particular, we have that any quasi-alternating knot gives rise to an infinite family of quasi-alternating links, in analogy with the behaviour of L-spaces. For example, it is well known that any sufficiently large surgery on a torus knot, or more generally a Berge knot, gives rise to an L-space [40].

Proposition 4.8. *For large enough integer surgery coefficient N, the branch set for $S^3_N(K)$ is quasi-alternating whenever K is a Berge knot. Moreover, for every $\frac{p}{q} > N$ the branch set associated to $\frac{p}{q}$-surgery on K must be quasi-alternating.*

Proof. For any Berge knot K there is some integer N, positive up to taking mirrors, with the property that $S^3_N(K)$ is a lens space [4]. As a result, $(\mu, N\mu+\lambda, (N+1)\mu+\lambda)$ gives a triad of slopes, in terms of the preferred basis (μ,λ) for K.

Moreover, since Berge knots are strongly invertible [36], there is an associated quotient tangle $T = (B^3,\tau)$ with representative so chosen so that $\tau(\frac{1}{0})$ is unknotted, and $S^3_N(K) = \mathbf{\Sigma}(S^3,\tau(0))$. By construction, both branch sets are quasi-alternating: the trivial knot $\tau(\frac{1}{0})$ and some non-split 2-bridge link $\tau(0)$ since $\mathbf{\Sigma}(S^3,\tau(0))$ is a lens space [20].

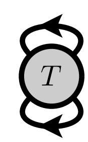
Now applying Theorem 4.7, $\tau(\frac{p}{q})$ must be quasi-alternating for every $\frac{p}{q} \geq 0$, so that the L-space $S^3_{(Nq+p)/q}(K)$ is branched over S^3 with quasi-alternating branch set $\tau(\frac{p}{q})$. $\square$

As a result, many – though not all – of the L-spaces arising as surgery on a Berge knot are also obtained as two-fold branched covers of quasi-alternating links. This implies in particular that the corresponding branch sets have thin Khovanov homology by work of Manolescu and Ozsváth [31]. Although this cannot be the case for all possible fillings when K is non-trivial (see Section 6.4), it turns out that in terms of homological width, the branch set corresponding to a filling of a Berge knot cannot be too much more complicated.

Proposition 4.9. *Surgery on a Berge knot has branch set with width at most 2.*

The proof of this statement (given in Section 4.6) is a consequence of a particular stable behaviour for branch sets associated to Dehn surgery, which we develop in the following sections.

4.3. **A mapping cone for integer surgeries.** Given a strongly invertible knot $K \hookrightarrow S^3$, with fixed strong inversion, let $T = (B^3,\tau)$ be the preferred representative of the associated quotient tangle. Therefore, $\tau(\frac{1}{0})$ is the trivial knot, and $S^3_0(K) \cong \mathbf{\Sigma}(S^3,\tau(0))$. As a result, $\widetilde{\mathrm{Kh}}(\tau(\frac{1}{0})) \cong \mathbb{F}$, and $w(\tau(0)) > 1$ since $\det(\tau(0)) = 0$. Notice that $\tau(0)$ is a two component link.



In the interest of studying the Khovanov homology of the branch sets associated to integer surgery, we choose the orientation on $\tau(0)$ shown on the right. That this is possible follows from the fact that $\tau(\frac{1}{0})$ is the trivial knot; that such a choice is copacetic results from the fact that $\widetilde{\mathrm{Kh}}(\tau(0))$, in the present context, is a relatively bi-graded group (only the absolute grading depends on orientation, as in Section 2, so we are free to fix any orientation for convenience). With this orientation on $\tau(0)$, there is a natural constant related to a fixed diagram for a representative of the associated quotient tangle $c_\tau = n_-(\tau(\frac{1}{0})) - n_-(\tau(0))$. Since $\tau(\frac{1}{0})$ has a single component, c_τ is independent of choice of orientation on $\tau(\frac{1}{0})$.

For example, we may rewrite the mapping cones in Khovanov homology as

$$\widetilde{\mathrm{Kh}}(\tau(1)) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(0))[-\tfrac{1}{2},\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[-\tfrac{c_\tau}{2},\tfrac{3c_\tau+2}{2}]\right)$$

since $c = n_-(\tau(\frac{1}{0})) - n_-(\tau(1)) = n_-(\tau(\frac{1}{0})) - n_-(\tau(0)) = c_\tau$, and

$$\widetilde{\mathrm{Kh}}(\tau(-1)) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[-\tfrac{c_\tau}{2},\tfrac{3c_\tau-2}{2}] \to \widetilde{\mathrm{Kh}}(\tau(0))[\tfrac{1}{2},-\tfrac{1}{2}]\right)$$

since $c = n_-(\tau(\frac{1}{0})) - n_-(\tau(-1)) = n_-(\tau(\frac{1}{0})) - n_-(\tau(0)) - 1 = c_\tau - 1$. Notice that in this case there is an overall $[1,0]$ shift (which may be ignored, as our interest is in the relative gradings and not the absolute gradings) so that

$$\widetilde{\mathrm{Kh}}(\tau(-1)) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[-\tfrac{c_\tau}{2},\tfrac{3c_\tau-2}{2}][-1,0] \to \widetilde{\mathrm{Kh}}(\tau(0))[-\tfrac{1}{2},-\tfrac{1}{2}]\right)[1,0]$$

which allows comparison of the homology of $\tau(\pm 1)$ in terms of $\widetilde{\mathrm{Kh}}(\tau(0))$ and the new generator $\widetilde{\mathrm{Kh}}(\tau(\frac{1}{0})) \cong \mathbb{F}$. More generally, we have:

Lemma 4.10. *For any integer m, and positive integer n,*

$$\widetilde{\mathrm{Kh}}(\tau(m+n)) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m)) \to \textstyle\bigoplus_n \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))\right)$$

as a relatively $\mathbb{Z}\oplus\mathbb{Z}$-graded group, where the integer m may be interpreted as a change of framing. More precisely, there exist explicit constants x and y and an identification

$$\textstyle\bigoplus_{q=0}^{n-1} \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x,y][0,q] \cong \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]$$

as graded $\mathbb{F}$-vector spaces so that

$$\widetilde{\mathrm{Kh}}(\tau(m+n)) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m)) \to \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]\right).$$

Proof. This amounts to careful iterated application of the mapping cone for resolution of a positive crossing applied to the n positive crossings in $\tau(m+n)$. When $n=1$ we have

$$\widetilde{\mathrm{Kh}}(\tau(m+1)) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m))[-\tfrac{1}{2},\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[-\tfrac{k_\tau}{2},\tfrac{3k_\tau+2}{2}]\right)$$

where $k_\tau = c_\tau + m$. Set $[x,y] = [-\frac{k_\tau}{2},\frac{3k_\tau+2}{2}]$. Now when $n=2$ we obtain

$$\begin{aligned}\widetilde{\mathrm{Kh}}(\tau(m+2)) &\cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m+1))[-\tfrac{1}{2},\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[-\tfrac{k_\tau+1}{2},\tfrac{3(k_\tau+1)+2}{2}]\right)\\ &\cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m+1))[-\tfrac{1}{2},\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x,y][-\tfrac{1}{2},\tfrac{1}{2}][0,1]\right)\end{aligned}$$

or, by unpacking the group $\widetilde{\mathrm{Kh}}(\tau(m+1))$ as in the previous case,

$$\begin{aligned}&\widetilde{\mathrm{Kh}}(\tau(m+2))\\ &\cong H_*\left(H_*\left(\widetilde{\mathrm{Kh}}(\tau(m))[-\tfrac{1}{2},\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x,y]\right)[-\tfrac{1}{2},\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x,y][-\tfrac{1}{2},\tfrac{1}{2}][0,1]\right)\end{aligned}$$

as an iterated mapping cone. Said another way, this expression is simply the repeated application of the long exact sequence. This simplifies considerably however, since the two occurrences of the group $\widetilde{\mathrm{Kh}}(\tau(\frac{1}{0})) \cong \mathbb{F}$ appear in the same δ-grading. Since the differential of the mapping cone (or,

the connecting homomorphism of the long exact sequence) raises δ-grading by one, there cannot be a differential between the copies of $\widetilde{\mathrm{Kh}}(\tau(\frac{1}{0}))$. As result,

$$\begin{aligned}
&\widetilde{\mathrm{Kh}}(\tau(m+2))\\
&\cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m))[-1,1] \to \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x,y][-\tfrac{1}{2},\tfrac{1}{2}] \oplus \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x,y][-\tfrac{1}{2},\tfrac{1}{2}][0,1]\right)\\
&\cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m))[-1,1] \to \textstyle\bigoplus_{q=0}^{1} \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x,y][-\tfrac{1}{2},\tfrac{1}{2}][0,q]\right)\\
&\cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m))[-\tfrac{1}{2},\tfrac{1}{2}] \to \textstyle\bigoplus_{q=0}^{1} \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x,y][0,q]\right)[-\tfrac{1}{2},\tfrac{1}{2}]
\end{aligned}$$

Now suppose for induction that

$$\widetilde{\mathrm{Kh}}(\tau(m+n-1)) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m))[-\tfrac{1}{2},\tfrac{1}{2}] \to \textstyle\bigoplus_{q=0}^{n-2} \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x,y][0,q]\right)[-\tfrac{n-2}{2},\tfrac{n-2}{2}]$$

and consider the group

$$\widetilde{\mathrm{Kh}}(\tau(m+n)) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m+n-1))[-\tfrac{1}{2},\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}(\tau(0))[-\tfrac{c}{2},\tfrac{3c+2}{2}]\right)$$

where $c = n_-(\tau(\frac{1}{0})) + n - 1 - n_-(\tau(m+n-1)) = n_-(\tau(\frac{1}{0})) - n_-(\tau(m)) + n - 1 = k_\tau + n - 1$. Then

$$\begin{aligned}
&\widetilde{\mathrm{Kh}}(\tau(m+n))\\
&\cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m+n-1))[-\tfrac{1}{2},\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}(\tau(0))[-\tfrac{k_\tau+n-1}{2},\tfrac{3(k_\tau+n-1)+2}{2}]\right)\\
&\cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m+n-1))[-\tfrac{1}{2},\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}(\tau(0))[-\tfrac{k_\tau}{2},\tfrac{3k_\tau+2}{2}][0,n-1][-\tfrac{n-1}{2},\tfrac{n-1}{2}]\right)\\
&\cong H_*\Bigg(H_*\left(\widetilde{\mathrm{Kh}}(\tau(m))[-\tfrac{1}{2},\tfrac{1}{2}] \to \textstyle\bigoplus_{q=0}^{n-2} \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x,y][0,q]\right)[-\tfrac{n-2}{2},\tfrac{n-2}{2}][-\tfrac{1}{2},\tfrac{1}{2}]\\
&\qquad\qquad \to \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x,y][0,n-1][-\tfrac{n-1}{2},\tfrac{n-1}{2}]\Bigg)\\
&\cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m))[-\tfrac{1}{2},\tfrac{1}{2}] \to \textstyle\bigoplus_{q=0}^{n-1} \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x,y][0,q]\right)[-\tfrac{n-1}{2},\tfrac{n-1}{2}]
\end{aligned}$$

noting once again that each of the occurrences of $\widetilde{\mathrm{Kh}}(\tau(\frac{1}{0}))$ differs only in the secondary grading.

Now as a relatively graded group, we are free to ignore the overall grading shift $[-\frac{n-1}{2},\frac{n-1}{2}]$. Moreover, since $\widetilde{\mathrm{Kh}}(\tau(\frac{1}{0})) \cong \mathbb{F}$, fixing an identification

$$\textstyle\bigoplus_{q=0}^{n-1} \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x+\tfrac{1}{2},y-\tfrac{1}{2}][0,q] \cong \mathbb{F}[q]/q^n \cong \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]$$

we have that

$$\widetilde{\mathrm{Kh}}(\tau(m+n)) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m)) \to \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]\right)$$

as a relatively $\mathbb{Z}\oplus\mathbb{Z}$-graded group. □

Remark 4.11. *As stated, this lemma might be viewed from the point of Heegaard-Floer homology. In particular, the long exact sequence for integer surgeries may be stated*

$$\cdots \longrightarrow \widehat{\mathrm{HF}}(S^3_m(K)) \longrightarrow \widehat{\mathrm{HF}}(S^3_{m+n}(K)) \longrightarrow \textstyle\bigoplus_n \widehat{\mathrm{HF}}(S^3) \longrightarrow \cdots$$

where

$$\textstyle\bigoplus_n \widehat{\mathrm{HF}}(S^3) \cong \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]$$

when viewed with twisted coefficients (c.f. [42, Theorem 3.1]*). We have given an analogous statement in terms of the Khovanov homology of the associated branch sets in the case when K is strongly invertible, a fact that is particularly interesting in light of* [41, Theorem 1.1] *relating Khovanov homology and Heegaard-Floer homology for two-fold branched covers by a spectral sequence.*

Before turning to consequences of Lemma 4.10, we note that a similar statement is forced to exist for negative surgeries. Indeed, consider $\widetilde{\mathrm{Kh}}(\tau(m-n))$ for any integer m, and positive integer n. Setting $m'=m-n$ we have that

$$\widetilde{\mathrm{Kh}}(\tau(m')) \cong \widetilde{\mathrm{Kh}}(\tau(m-n))$$

and

$$\begin{aligned}\widetilde{\mathrm{Kh}}(\tau(m)) &\cong \widetilde{\mathrm{Kh}}(\tau(m'+n))\\ &\cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m')) \to \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]\right)\\ &\cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m-n)) \to \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]\right).\end{aligned}$$

It follows that:

Lemma 4.12. *For any integer m, and positive integer n,*

$$\widetilde{\mathrm{Kh}}(\tau(m-n)) \cong H_*\left(\bigoplus_n \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0})) \to \widetilde{\mathrm{Kh}}(\tau(m))\right)$$

as a relatively $\mathbb{Z}\oplus\mathbb{Z}$-graded group, where the integer m may be interpreted as a change of framing. More precisely, there exist an explicit constants x' and y' (different than above) and an identification

$$\bigoplus_{q=0}^{n-1} \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x',y'][0,q] \cong \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]$$

so that

$$\widetilde{\mathrm{Kh}}(\tau(m-n)) \cong H_*\left(\mathbb{F}[\mathbb{Z}/n\mathbb{Z}] \to \widetilde{\mathrm{Kh}}(\tau(m))\right)$$

Remark 4.13. *In fact, it should be immediately clear that in this case the group*

$$\bigoplus_{q=0}^{n-1} \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x',y'][0,q] \cong \mathbb{F}[q^{-1}]/q^{-n} \cong \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]$$

must lie in grading $\delta-1$ relative to the group

$$\bigoplus_{q=0}^{n-1} \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x+\tfrac{1}{2},y-\tfrac{1}{2}][0,q] \cong \mathbb{F}[q]/q^{n} \cong \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]$$

of Lemma 4.10 in grading δ. Alternatively, Lemma 4.12 may be proved directly by an argument nearly identical to the argument of Lemma 4.10, up to renaming constants.

4.4. **Width stability.** There are two essential consequences that we derive from Lemma 4.10. Similar properties exist for branch sets associated to negative surgeries, and theses may be easily inferred by the reader. We will not state these, opting instead to pass to positive surgeries on the mirror to avoid negative coefficients.

Lemma 4.14. *For $N >> 0$ the exact sequence for $\widetilde{\mathrm{Kh}}(\tau(N+1))$ splits so that, ignoring gradings,*

$$\widetilde{\mathrm{Kh}}(\tau(N+1)) \cong \widetilde{\mathrm{Kh}}(\tau(N)) \oplus \mathbb{F}.$$

Proof. Let $N = m$ and $n = 1$ in the notation of Lemma 4.10, so that

$$\widetilde{\mathrm{Kh}}(\tau(N+1)) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(N)) \to \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))\right)$$

On the other hand, with $m = 0$ and $n = N + 1$ we have that

$$\widetilde{\mathrm{Kh}}(\tau(N+1)) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(0)) \to \mathbb{F}[q]/q^{N+1}\right).$$

Since the differential preserves the secondary q-grading, for $N >> 0$ the generator represented by q^N cannot be in the image of the differential. □

Lemma 4.15. *Up to overall shift the generators* $\widetilde{\mathrm{Kh}}(\tau(\frac{1}{0})) \cong \mathbb{F}$, *when they survive in homology, are all supported in a single relative* δ*-grading.*

Proof. Immediate from the identification with the truncated polynomial ring in Lemma 4.10. □

As a result of Lemma 4.14, the width of the $\tau(n)$ may be calculated for all n once some finite collection of the values is known. Moreover, these quantities must be bounded, in light of Lemma 4.15.

Definition 4.16. *For a given strongly invertible knot and preferred associated quotient tangle, define* $w_{\max} = \max_{n\in\mathbb{Z}}\{w(\tau(n))\}$ *and* $w_{\min} = \min_{n\in\mathbb{Z}}\{w(\tau(n))\}$.

Lemma 4.17. *Suppose* $w_{\min} = w(\tau(N))$ *for* $|N| >> 0$*. Then either* $w_{\min} = 1$ *and* $T = (B^3, \tau)$ *is the tangle associated to the trivial knot, or* $w_{\min} > 1$ *in which case* $w_{\min} = w_{\max}$.

Proof. First suppose $w_{\min} = 1$, so that $w(\tau(N)) = 1$ for all $|N|$ sufficiently large. Then by Proposition 4.2, $S^3_{\pm N}(K) = \mathbf{\Sigma}(S^3, \tau(\pm N))$ must be an L-space for all N sufficiently large. However, if $S^3_N(K)$ is an L-space, for N large enough in absolute value, then K is the trivial knot.

To see this, note that since $S^3_N(K)$ is an L-space for $N >> 0$ we have that $g(K) = |\boldsymbol{\tau}(K)|$, where $\boldsymbol{\tau}(K)$ is the Ozsváth-Szabó concordance invariant of K, by [40, Proposition 3.3]. On the other hand, $S^3_{-N}(K) \cong -S^3_N(K^\star)$ is an L-space as well, so that $g(K^\star) = |\boldsymbol{\tau}(K^\star)|$. However, it is a standard property of $\boldsymbol{\tau}$ that $\boldsymbol{\tau}(K^\star) = -\boldsymbol{\tau}(K)$ [37, Lemma 3.3]. Therefore, since $g(K) = g(K^\star)$ we have shown that $\boldsymbol{\tau}(K) = g(K) = -\boldsymbol{\tau}(K)$ hence $g(K) = 0$ and K must be the trivial knot (thus $\tau(0) \simeq \bigcirc \sqcup \bigcirc$ with width 2).

Now suppose that $w_{\min} = \tau(N) > 1$ for $|N| >> 0$, and choose $m << 0$ so that $w(\tau(m)) = w_{\min}$. Then we have

$$\widetilde{\mathrm{Kh}}(\tau(m+n)) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m)) \to \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]\right)$$

for all $n > 0$. In particular, since $w_{\min} = w(\tau(m+n))$ for n sufficiently large, it must be that $\mathrm{Supp}(\mathbb{F}[\mathbb{Z}/n\mathbb{Z}]) \subset \mathrm{Supp}(\widetilde{\mathrm{Kh}}(\tau(m))$ (and the width can not increase). On the other hand, a decrease in width would contradict the assumption that $w(\tau(m))$ is minimal, hence $w_{\min} = w_{\max}$. □

Lemma 4.18. *The maximum and minimum widths differ by at most 1: either* $w_{\max} = w_{\min}$ *or* $w_{\max} = w_{\min} + 1$.

Proof. First notice that the statement holds for the tangle associated to the quotient of the trivial knot by Lemma 4.17, since $w(\tau(0)) = w(\bigcirc \sqcup \bigcirc) = 2$ and $\tau(n) = 1$ for all $n \neq 0$.

If K is non trivial, without loss of generality we may suppose that $w_{\min} = w(\tau(N))$ for $N >> 0$ and that $w_{\max} = w(\tau(N))$ for $N << 0$. Now choosing $m << 0$, in the notation of Lemma 4.10 we have that $w_{\max} = w(\tau(m))$. Further,

$$\widetilde{\mathrm{Kh}}(\tau(m+n)) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m)) \to \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]\right)$$

for every $n > 0$. Since $w_{\min} = w(\tau(m+n))$ for some n, the group $\mathbb{F}^n \cong \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]$ must be in a fixed grading supported by $\widetilde{\mathrm{Kh}}(\tau(m))$. Therefore, if

$$\widetilde{\mathrm{Kh}}(\tau(m)) \cong \textstyle\bigoplus_\delta \mathbb{F}^{b_\delta} \cong \mathbb{F}^{b_1} \oplus \mathbb{F}^{b_2} \oplus \cdots \oplus \mathbb{F}^{b_{w_{\max}}}$$

then since the differential of the mapping cone raises δ-grading by one we have that $w_{\max} = w_{\min}$ unless

$$\widetilde{\mathrm{Kh}}(\tau(m+n)) \cong H_* \begin{pmatrix} \mathbb{F}^{b_1} & & \mathbb{F}^{b_2} & \cdots & \mathbb{F}^{b_{w_{\max}}} \\ & \searrow & & & \\ & & \mathbb{F}^n & & \end{pmatrix}$$

wherein the possibility arises for $w_{\max} = w_{\min} + 1$. □

Remark 4.19. *It follows from the above argument that, whenever $w_{\max} = w_{\min} + 1$ for a tangle associated to a non-trivial knot in S^3, there is a unique ℓ for which $w(\tau(\ell))$ and $w(\tau(\ell+1))$ differ. Moreover, we may assume up to taking mirrors that $\ell \geq 0$.*

We note that, having fixed $\ell \geq 0$ whenever $w_{\max} = w_{\min} + 1$, the width either *expands* or *decays*. More precisely, for $m = 0$ in the notation of Lemma 4.10, the width expands whenever

$$\widetilde{\mathrm{Kh}}(\tau(n)) \cong H_* \begin{pmatrix} \mathbb{F}^{b_1} & \mathbb{F}^{b_2} & \cdots & \mathbb{F}^{b_{w_{\min}}} & & \\ & & & & \searrow & \\ & & & & & \mathbb{F}^n \end{pmatrix}$$

and the possibility for width decay arises whenever

$$\widetilde{\mathrm{Kh}}(\tau(n)) \cong H_* \begin{pmatrix} \mathbb{F}^{b_1} & & \mathbb{F}^{b_2} & \cdots & \mathbb{F}^{b_{w_{\max}}} \\ & \searrow & & & \\ & & \mathbb{F}^n & & \end{pmatrix}$$

For example, Berge knots (chosen so that the lens space surgeries are positive) give rise to a family of tangles for which the width decays (c.f. Proposition 4.9).

4.5. **On determinants and resolutions.** In the arguments that follow, we will rely heavily on resolutions of terminal crossings (see Definition 3.11) in branch sets $\tau(\frac{p}{q})$ for which $S^3_{p/q}(K) = \mathbf{\Sigma}(B^3, \tau(\frac{p}{q}))$. As such, we remark that $\det(\tau(\frac{p}{q})) = |H_1(S^3_{p/q}(K);\mathbb{Z})| = p$ for any $\frac{p}{q} \geq 0$ (in all cases, we deal with negative surgeries by passing to the mirror image). Moreover if $\tau(\frac{p_0}{q_0})$ and $\tau(\frac{p_1}{q_1})$ are the links resulting from resolution of the terminal crossing, then

$$\det(\tau(\tfrac{p}{q})) = p = p_0 + p_1 = \det(\tau(\tfrac{p_0}{q_0})) + \det(\tau(\tfrac{p_1}{q_1}))$$

by applying Property 3.10.

As a result, $\widetilde{\mathrm{Kh}}(\tau(\frac{p}{q}))$ may be studied by applying Proposition 2.6 to the resolutions $\tau(\frac{p_0}{q_0})$ and $\tau(\frac{p_1}{q_1})$ whenever $\frac{p}{q} > 1$. In the case $\frac{p}{q} \in (0,1)$ the same arguments work by using Proposition 2.7 when treating continued fractions of length $r = 2$: here $\det(\tau(\frac{p_0}{q_0})) = \det(\tau(0)) = 0$.

By Lemma 4.10 we have that

$$\widetilde{\mathrm{Kh}}(\tau(n)) \cong H_* \left(\widetilde{\mathrm{Kh}}(\tau(0)) \to \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]\right)$$

for a specific identification of $\bigoplus_n \widetilde{\mathrm{Kh}}(\tau(\frac{1}{0})) \cong \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]$ as a graded group. As a result,

$$\widetilde{\mathrm{Kh}}_\sigma(\tau(n)) \cong H_* \left(\widetilde{\mathrm{Kh}}_\sigma(\tau(n-1)) \to \widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{1}{0}))\right)$$

whenever $n > 1$, and

$$\widetilde{\mathrm{Kh}}_\sigma(\tau(1)) \cong H_* \left(\widetilde{\mathrm{Kh}}(\tau(0))[-\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{1}{0}))\right)$$

where $\widetilde{\mathrm{Kh}}_\sigma(\tau(\frac{1}{0})) = \widetilde{\mathrm{Kh}}(\tau(\frac{1}{0}))$ since the signature of the trivial knot is 0. In either case,

$$\mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(n))\right) \subseteq \mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(n+1))\right)$$

or

$$\mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(n+1))\right) \subseteq \mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(n))\right)$$

as absolutely $\mathbb{Z}$-graded groups (where the fixed shifts are adjusted accordingly by $[-\frac{1}{2}]$ in the case $n = 0$). Notice that these inclusions are equalities whenever $w(\tau(n)) = w(\tau(n+1))$, so that the inclusions are only relevant in the case when the width changes by one.

4.6. **An upper bound for width.**

Proposition 4.20. *Let K be a strongly invertible knot in S^3, with preferred associated quotient tangle $T = (B^3, \tau)$. Then $w(\tau(\frac{p}{q}))$ is bounded above by $w_{\max}$ for all $\frac{p}{q} \in \mathbb{Q}$.*

Proof. By taking mirrors, we suppose without loss of generality that $\frac{p}{q} \geq 0$ and proceed in 2 cases.

Case 1: $1 \leq \frac{p}{q}$

By its definition, $w_{\max}$ provides the upper bound for $w(\tau(n))$ for any n. This provides a base for induction in r, the length of the continued fraction representation $\frac{p}{q} = [a_1, a_2, \ldots, a_r]$.

First consider the case $\frac{p}{q} = [a_1, 2]$. Here we have

$$\det(\tfrac{p}{q}) = p = p_0 + p_1 = a_1 + a_1 + 1 = \det(\tfrac{p_0}{q_0}) + \det(\tfrac{p_1}{q_1})$$

where $\frac{p_0}{q_0} = [a_1]$ and $\frac{p_1}{q_1} = [a_1, 1] = [a_1 + 1]$ by resolving the terminal crossing. In either case $w(\frac{p_0}{q_0}), w(\frac{p_1}{q_1}) \leq w_{\max}$, and by applying Proposition 2.6 we have

$$\widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p}{q})) \cong H_* \left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_0}{q_0})) \to \widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_1}{q_1}))\right).$$

Moreover, according to Section 4.5 we have that either

$$\mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_1}{q_1}))\right) \subseteq \mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_0}{q_0}))\right)$$

or

$$\mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_0}{q_0}))\right) \subseteq \mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_1}{q_1}))\right)$$

(depending on expansion or decay) as a consequence of Lemma 4.10. Therefore,

$$\begin{aligned} w(\tau[a_1,2]) &= w(\tau(\tfrac{2a_1+1}{2})) \\ &\le \max\{w(\tau\lfloor\tfrac{2a_1+1}{2}\rfloor), w(\tau\lceil\tfrac{2a_1+1}{2}\rceil)\} \\ &= \max\{w(\tau(a_1)), w(\tau(a_1+1))\} \\ &\le w_{\max}. \end{aligned}$$

The same statement holds for $\frac{p}{q} = [a_1, a_2]$. By iterating Proposition 2.6 we have

$$\begin{tikzcd} \widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_1}{q_1})) \arrow[r] & \widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p}{q})) \arrow[d] \\ \widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_1}{q_1})) \arrow[r] & \widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_0'}{q_0'})) \arrow[d] \\ & \vdots \arrow[d] \\ & \widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_0}{q_0})) \end{tikzcd}$$

where the connecting homomorphisms have been omitted (of course, there is no danger in doing so since these can only decrease the homological width). Once again, as a consequence of supports we conclude that $w(\tau[a_1,a_2]) \le \max\{w(\tau(a_1)), w(\tau(a_1+1))\} \le w_{\max}$.

Now for induction in r: given $\frac{p}{q} = [a_1, a_2, \ldots, a_{r-1}]$ the inductive hypothesis is that $w(\tau(\frac{p}{q})) \le w_{\max}$ and one of

$$\mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau[a_1,a_2,\ldots,a_{r-1}])\right) \subseteq \mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau[a_1,a_2,\ldots,a_{r-1}+1]))\right)$$

or

$$\mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau[a_1,a_2,\ldots,a_{r-1}+1]))\right) \subseteq \mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau[a_1,a_2,\ldots,a_{r-1}])\right)$$

holds.

This being the case, we claim that

$$w(\tau[a_1,a_2,\ldots,a_{r-1},a_r]) \le \max\left\{w(\tau[a_1,a_2,\ldots,a_{r-1}]), w(\tau[a_1,a_2,\ldots,a_{r-1}+1])\right\}.$$

By resolving the terminal crossing of $\tau(\frac{p}{q})$ and applying Proposition 2.6

$$\widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p}{q})) \cong H_*\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_0}{q_0})) \to \widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_1}{q_1}))\right)$$

so that $w(\tau(\frac{p}{q})) \le \max\{w(\tau(\frac{p_0}{q_0})), w(\tau(\frac{p_1}{q_1}))\}$ if $a_r = 2$. By induction in a_r we have that

$$w(\tau[a_1,a_2,\ldots,a_{r-1},a_r]) \le \max\{w(\tau[a_1,a_2,\ldots,a_{r+1}]), w(\tau[a_1,a_2,\ldots,a_{r-1}+1])\}$$

by applying Property 3.10 together with the induction hypothesis on supports.

As a result, by induction in length we have that $w(\tau(\frac{p}{q})) \leq \{w(\tau\lfloor\frac{p}{q}\rfloor), w(\tau\lceil\frac{p}{q}\rceil)\} \leq w_{\max}$, concluding the proof in this case.

Case 2: $0 < \frac{p}{q} < 1$

The proof in this case follows the same lines as the previous case, and differs only in passing from the case $r = 2$ to $r = 1$. Indeed, the argument here is identical, once we replace the use of Proposition 2.6 is with that of its degenerative counterpart, Proposition 2.7. This is due to the fact that, while the determinants remain additive under resolution, $\det(\tau\lfloor\frac{p}{q}\rfloor) = 0$ in this case.

To see that this is so, consider once again the case $\frac{p}{q} = [a_1, 2] = [0, 2]$. By applying Proposition 2.7 we have

$$\widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p}{q})) \cong H_*\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_0}{q_0}))[-\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_1}{q_1}))\right).$$

Moreover, according to Section 4.5 we have that either

$$\mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_1}{q_1}))[-\tfrac{1}{2}]\right) \subseteq \mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_0}{q_0}))\right)$$

or

$$\mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_0}{q_0}))\right) \subseteq \mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_1}{q_1}))[-\tfrac{1}{2}]\right)$$

as a consequence of Lemma 4.10. Therefore,

$$\begin{aligned} w(\tau[a_1, 2]) &= w(\tau(\tfrac{2a_1+1}{2})) \\ &\leq \max\{w(\tau\lfloor\tfrac{2a_1+1}{2}\rfloor), w(\tau\lceil\tfrac{2a_0+1}{2}\rceil)\} \\ &= \max\{w(\tau(a_1)), w(\tau(a_1+1))\} \\ &\leq w_{\max}. \end{aligned}$$

The same statement holds for $\frac{p}{q} = [a_1, a_2]$. By iterating Proposition 2.7 as in the previous case so that $w(\tau[a_1, a_2]) \leq \max\{w(\tau(a_1)), w(\tau(a_1+1))\} \leq w_{\max}$. □

Remark 4.21. *Case 2, when $\frac{p}{q} \in (0, 1)$, will be present in many of the arguments that follow. However, in every setting this case simply amounts to replacing Proposition 2.6 with Proposition 2.7 in passing from half-integer (continued fractions of length 2) to integer surgeries, as in the above proof. Thus we will restrict, without loss of generality, to the case $\frac{p}{q} \geq 1$ in the arguments below.*

With this upper bound in hand, we may now prove Proposition 4.9.

Proof of Proposition 4.9. For any Berge knot K, there is some N, positive up to taking mirrors, for which $S^3_N(K)$ is a lens space. By a result of Osborne [36], K is a strongly invertible knot, so let $T = (B^3, \tau)$ be a representative for the associated quotient tangle compatible with the basis for surgery $(\mu, N\mu + \lambda)$, where λ is the preferred longitude. Note that $S^3_N(K) \cong \mathbf{\Sigma}(S^3, \tau(0))$ in this setting.

Now we have already seen that $w(\tau(n)) = 1$ for all $n \geq 0$ as a result of Proposition 4.8, since quasi-alternating knots are thin by a result of Manolescu and Ozsváth [31]. On the other hand,

$w(\tau(n))$ is at most 2 when $n<0$ in application of Lemma 4.18. Therefore, for any Berge knot, the associated quotient tangle has $w_{\min}=1$ and $w_{\max}=2$.

The result now follows from an application of Proposition 4.20: $w(\tau(\frac{p}{q}))$ is bounded above by $w_{\max}=2$. $\square$

4.7. **A lower bound for width.**

Proposition 4.22. *Let K be a strongly invertible knot in S^3, with preferred associated quotient tangle $T=(B^3,\tau)$. If $w_{\max}=w_{\min}$ then $w(\tau(\frac{p}{q}))$ is bounded below by $w_{\min}$ for all $\frac{p}{q}\in\mathbb{Q}$.*

Proof. Without loss of generality, assume that $\frac{p}{q}\geq 1$.

Since $w_{\max}=w_{\min}=w$, we have that $w=w(\tau(n))$ for every $n\in\mathbb{Z}$. In particular,

$$\operatorname{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(n+1))\right)=\operatorname{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(n))\right)$$

as a consequence of Lemma 4.10. Thus, applying Proposition 2.6

$$\widetilde{\mathrm{Kh}}_\sigma(\tau[a_1,2])\cong H_*\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(a_1))\to\widetilde{\mathrm{Kh}}_\sigma(\tau(a_1+1))\right)$$

so that if $\widetilde{\mathrm{Kh}}(\tau(a_1))\cong\mathbb{F}^{b_1}\oplus\cdots\oplus\mathbb{F}^{b_w}$ and $\widetilde{\mathrm{Kh}}(\tau(a_1+1))\cong\mathbb{F}^{b'_1}\oplus\cdots\oplus\mathbb{F}^{b'_w}$ (note that $b_i\neq b'_i$ for precisely one value $1\leq i\leq w$) then

$$\widetilde{\mathrm{Kh}}(\tau[a_1,2])\cong H_*\begin{pmatrix}\mathbb{F}^{b_1} & & \mathbb{F}^{b_2} & & \cdots & & \mathbb{F}^{b_w}\\ & \searrow & & \searrow & & \searrow & \\ \mathbb{F}^{b'_1} & & \mathbb{F}^{b'_2} & & \cdots & & \mathbb{F}^{b'_w}\end{pmatrix}$$

as a relatively graded group, since the differential of the mapping cone raises δ-grading by 1. Notice in particular that $b^*_1\geq b'_1$ and $b^*_w\geq b_w$ for $\widetilde{\mathrm{Kh}}(\tau[a_1,2])=\mathbb{F}^{b^*_1}\oplus\cdots\oplus\mathbb{F}^{b^*_w}$, so that $w(\tau[a_1,2])=w$.

Similarly, for $\frac{p}{q}=[a_1,a_2]$ in general, we may iteratively apply Proposition 2.6 a_2-1 times to the same end:

$$\widetilde{\mathrm{Kh}}([a_1,a_2])\cong H_*\left(\begin{array}{c}\mathbb{F}^{b_1}\quad\mathbb{F}^{b_2}\quad\cdots\quad\mathbb{F}^{b_w}\\ \vdots\qquad\vdots\qquad\qquad\vdots\\ H_*\begin{pmatrix}\mathbb{F}^{b_1} & & \mathbb{F}^{b_2} & & \cdots & & \mathbb{F}^{b_w}\\ & \searrow & & \searrow & & \searrow & \\ \mathbb{F}^{b'_1} & & \mathbb{F}^{b'_2} & & \cdots & & \mathbb{F}^{b'_w}\end{pmatrix}\end{array}\right)$$

so that $b^*_1\geq b'_1$ and $b^*_w\geq b_w$ for $\widetilde{\mathrm{Kh}}(\tau[a_1,a_2])=\mathbb{F}^{b^*_1}\oplus\cdots\oplus\mathbb{F}^{b^*_w}$, and once again $w(\tau[a_1,a_2])=w$.

To complete the proof then, we induct in r with the assumption that $w(\tau(\frac{p}{q}))=w$ for all $\frac{p}{q}=[a_1,\ldots,a_{r-1}]$, and that

$$\operatorname{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau[a_1,a_2,\ldots,a_{r-1}])\right)=\operatorname{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau[a_1,a_2,\ldots,a_{r-1}+1]))\right)$$

holds.

This being the case, we claim that

$$w(\tau[a_1, a_2, \dots, a_{r-1}, a_r]) \geq \min\left\{w(\tau[a_1, a_2, \dots, a_{r-1}]), w(\tau[a_1, a_2, \dots, a_{r-1}+1])\right\}.$$

Indeed, when $a_r = 2$ we have that

$$\widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p}{q})) \cong H_*\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_0}{q_0})) \to \widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_1}{q_1}))\right)$$

by applying Proposition 2.6 so that in either case $w(\tau(\frac{p}{q})) = w(\tau(\frac{p_0}{q_0})), w(\tau(\frac{p_1}{q_1}))$ since the corresponding groups have the same support. By induction in a_r we have that

$$w(\tau[a_1, a_2, \dots, a_{r-1}, a_r]) = w(\tau[a_1, a_2, \dots, a_{r+1}]), w(\tau[a_1, a_2, \dots, a_{r-1}+1])$$

as before, by applying the induction hypothesis on supports.

As a result, by induction in length we have that $w(\tau(\frac{p}{q})) = w$, concluding the proof. □

Combining Proposition 4.22 with Proposition 4.20 we have immediately that $w(\tau(-)) : \mathbb{Q} \to \mathbb{N}$ takes a single value $w \in \mathbb{N}$ when $w = w_{\max} = w_{\min}$, where $T = (B^3, \tau)$ is the preferred representative for the quotient tangle associated to a strongly invertible knot in S^3.

4.8. **Expansion and decay.** By Remark 4.19, if $w_{\max} = w_{\min} + 1$ then there is a unique value ℓ, which we may assume is positive, for which either $w_{\min} = w(\tau(\ell)) < w(\tau(\ell+1)) = w_{\max}$ (width *expansion*) or $w_{\max} = w(\tau(\ell)) > w(\tau(\ell+1)) = w_{\min}$ (width *decay*).

In each setting, we establish a sufficient condition for which $w_{\min}$ still provides a lower bound for $w(\tau(\frac{p}{q}))$.

Definition 4.23. *T is expansion generic if $b_k > 1$ where*

$$\widetilde{\mathrm{Kh}}(\tau(\ell)) \cong \mathbb{F}^{b_1} \oplus \cdots \oplus \mathbb{F}^{b_k}$$

so that $w_{\min} = k$ and

$$\widetilde{\mathrm{Kh}}(\tau(\ell+1)) \cong \mathbb{F}^{b_1} \oplus \cdots \oplus \mathbb{F}^{b_k} \oplus \mathbb{F}$$

so that $w_{\max} = k+1$, where $k > 0$.

Definition 4.24. *T is decay generic if $b_1 > 1$ where*

$$\widetilde{\mathrm{Kh}}(\tau(\ell)) \cong \mathbb{F} \oplus \mathbb{F}^{b_1} \oplus \cdots \oplus \mathbb{F}^{b_k}$$

so that $w_{\max} = k+1$ and

$$\widetilde{\mathrm{Kh}}(\tau(\ell+1)) \cong \mathbb{F}^{b_1} \oplus \cdots \oplus \mathbb{F}^{b_k}$$

so that $w_{\min} = k$, where $k > 0$.

Both of these notions are well defined, according Lemma 4.18. Notice that if T is expansion generic, then $T^\star$ is decay generic, and vice versa. These both seem to be stronger conditions than necessary, however genericity (in each sense) turns out to be the rule rather than the exception when we turn to applications of homological width.

Proposition 4.25. *If T is expansion generic then $w(\tau(\frac{p}{q}))$ is bounded below by $w_{\min}$ for all $\frac{p}{q} \in \mathbb{Q}$.*

Proof. Let $w(\tau(\ell)) = k = w_{\min}$ and $w(\tau(\ell+1)) = k+1 = w_{\max}$. First notice that for $\frac{p}{q} \notin [\ell, \ell+1]$ the proof proceeds exactly as in the proof of Proposition 4.22. Thus we are left to consider the case when $\frac{p}{q} \in [\ell, \ell+1]$. Without loss of generality, we may assume that $\ell > 0$: if this is not the case, the argument below goes through with Proposition 2.7 replacing Proposition 2.6 where necessary, as is now familiar.

Now when $[a_1, a_2] = [\ell, 2]$, we have that

$$\widetilde{\mathrm{Kh}}_\sigma(\tau[\ell,2]) \cong H_*\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\ell)) \to \widetilde{\mathrm{Kh}}_\sigma(\tau(\ell+1))\right)$$

by resolving the terminal crossing and applying Proposition 2.6. By applying Lemma 4.10, notice that

$$\mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\ell))\right) \subseteq \mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\ell+1))\right)$$

gives

$$\widetilde{\mathrm{Kh}}(\tau[\ell,2]) \cong H_*\left(\begin{tikzcd} \mathbb{F}^{b_1} \arrow[dr] & \mathbb{F}^{b_2} \arrow[dr] & \cdots \arrow[dr] & \mathbb{F}^{b_k} \arrow[dr] & \\ \mathbb{F}^{b_1} & \mathbb{F}^{b_2} & \cdots & \mathbb{F}^{b_k} & \mathbb{F} \end{tikzcd}\right)$$

so that $w(\tau[\ell,2]) \geq k$ due to expansion genericity ($b_k > 1$), since this ensures that groups in gradings 1 and k survive in homology.

Now consider the case $\frac{p}{q} = [\ell, 3]$. Again, we have that

$$\begin{aligned} \widetilde{\mathrm{Kh}}_\sigma(\tau[\ell,3]) &\cong H_*\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\ell)) \to \widetilde{\mathrm{Kh}}_\sigma(\tau[\ell,2])\right) \\ &\cong H_*\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\ell)) \to H_*\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\ell)) \to \widetilde{\mathrm{Kh}}_\sigma(\tau(\ell+1))\right)\right) \end{aligned}$$

so that

$$\widetilde{\mathrm{Kh}}(\tau[\ell,3]) \cong H_*\left(\begin{tikzcd} \mathbb{F}^{b_1} \arrow[dr] & \mathbb{F}^{b_2} \arrow[dr] & \cdots \arrow[dr] & \mathbb{F}^{b_k} \arrow[dr] & \\ \mathbb{F}^{b'_1} & \mathbb{F}^{b'_2} & \cdots & \mathbb{F}^{b'_k} & \mathbb{F}^{\epsilon} \end{tikzcd}\right)$$

where $\epsilon = 0, 1$ arising from

$$\widetilde{\mathrm{Kh}}(\tau[\ell,2]) \cong \mathbb{F}^{b'_1} \oplus \mathbb{F}^{b'_2} \oplus \cdots \oplus \mathbb{F}^{b'_k} \oplus \mathbb{F}^{\epsilon}.$$

Note that $b'_k > 0$, since $w(\tau[\ell,2]) \geq k$. If $\epsilon = 0$ then groups survive in degrees 1 and k so the width is k; in the case $\epsilon = 1$, $w(\tau[\ell,3]) \geq k$ due to expansion genericity as before. Proceeding in this way by iterating Proposition 2.6, we obtain the desired result for all $\tau(\frac{p}{q})$ when $\frac{p}{q} = [\ell, a_2]$. Notice that either

$$\mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau[\ell,a_2])\right) = \mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau[\ell,a_2+1])\right),$$

in which case the proof concludes along the lines of the proof of Proposition 4.22, or

$$\mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau[\ell,a_2])\right) \subseteq \mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau[\ell,a_2+1])\right).$$

In the case of the latter, we remark that $\widetilde{\mathrm{Kh}}(\tau[\ell,a_2]) \cong \mathbb{F}^{b_1} \oplus \cdots \oplus \mathbb{F}^{b_k}$ and $\widetilde{\mathrm{Kh}}(\tau[\ell,a_2]) \cong \mathbb{F}^{b'_1} \oplus \cdots \oplus \mathbb{F}^{b'_k} \oplus \mathbb{F}^{b'_{k+1}}$with $b_k > b'_{k+1}$.

We now proceed by induction, assuming the result holds for continued fractions of length $r-1$, with the support the Khovanov homology of the zero resolution of the terminal crossing included in the support of the Khovanov homology of the one resolution (with the gradings shifted by the signatures, according to Proposition 2.6).

Now for $\frac{p}{q} = [\ell, a_2, \ldots, a_{r-1}, 2]$,

$$\widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p}{q})) \cong H_*\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_0}{q_0})) \to \widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_1}{q_1}))\right)$$

so that

$$\widetilde{\mathrm{Kh}}(\tau(\tfrac{p}{q})) \cong H_*\left(\begin{array}{ccccccccc} \mathbb{F}^{b_1} & & \mathbb{F}^{b_2} & & \cdots & & \mathbb{F}^{b_k} & & \\ & \searrow & & \searrow & & \searrow & & \searrow & \\ \mathbb{F}^{b'_1} & & \mathbb{F}^{b'_2} & & \cdots & & \mathbb{F}^{b'_k} & & \mathbb{F}^{b'_{k+1}} \end{array}\right)$$

where $b_k > b'_{k+1}$. Therefore, since there must be non-trivial groups in the first and k^{th} gradings, $w(\tau(\frac{p}{q})) \geq k$. To conclude the proof then it remains only to iterate this argument in a_r, as in the case $r=2$. ◻

Proposition 4.26. *If T is decay generic then $w(\tau(\frac{p}{q}))$ is bounded below by $w_{\min}$ for all $\frac{p}{q} \in \mathbb{Q}$.*

Proof. The proof is almost identical to the proof of Proposition 4.25.

Let $w(\tau(\ell)) = k+1 = w_{\max}$ and $w(\tau(\ell+1)) = k = w_{\min}$. Again, notice that for $\frac{p}{q} \notin [\ell, \ell+1]$ the proof proceeds exactly as in the proof of Proposition 4.22. Thus we are left to consider the case when $\frac{p}{q} \in [\ell, \ell+1]$. Without loss of generality, we may assume that $\ell > 0$.

Now when $[a_1, a_2] = [\ell, 2]$ is a half-integer, we have that

$$\widetilde{\mathrm{Kh}}_\sigma(\tau[\ell,2]) \cong H_*\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\ell)) \to \widetilde{\mathrm{Kh}}_\sigma(\tau(\ell+1))\right)$$

by resolving the terminal crossing and applying Proposition 2.6. Notice however that since

$$\mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\ell+1))\right) \subseteq \mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\ell))\right)$$

this gives

$$\widetilde{\mathrm{Kh}}(\tau[\ell,2]) \cong H_*\left(\begin{array}{ccccccccc} \mathbb{F} & & \mathbb{F}^{b_1} & & \mathbb{F}^{b_2} & & \cdots & & \mathbb{F}^{b_k} \\ & \searrow & & \searrow & & \searrow & & \searrow & \\ & & \mathbb{F}^{b_1} & & \mathbb{F}^{b_2} & & \cdots & & \mathbb{F}^{b_k} \end{array}\right)$$

so that $w(\tau[\ell,2]) \geq k$ due to expansion genericity ($b_1 > 1$), since this ensures that groups in gradings 1 and k survive in homology.

The conclusion then follows by induction in the length of the continued fraction associated to $\frac{p}{q}$, assuming the inclusion of supports as before. ◻

Collecting the above results, we have that $w(\tau(-)) : \mathbb{Q} \to \mathbb{N}$ takes values $[w_{\min}, w_{\min}+1] \subset \mathbb{N}$ when in the decay or expansion generic setting, where $T = (B^3, \tau)$ is the preferred representative for the quotient tangle associated to a strongly invertible knot in S^3.

5. Surgery obstructions

5.1. **Width bounds for lens spaces.** Combining work of Hodgson and Rubinstein [20] with work of Lee [29], we have the following statement:

Theorem 5.1 (Hodgson-Rubinstein [20], Lee [29])**.** *If Y is a lens space, then Y is a two-fold branched cover of S^3, with branch set of width 1.*

Note that this excludes the manifold $S^2 \times S^1$ since it is branched over the 2-component trivial link having width 2.

Proof of Theorem 5.1. By work of Hodgson and Rubinstein, only non-split two-bridge links arise as the branch sets of lens spaces [20]. As a result, to generate this collection of branch sets we need to consider surgery on the trivial knot in S^3; the associated quotient tangle is rational, and the preferred representative is $(B^3, \asymp)$ since $\det(\tau(0)) = \det(\bigcirc \sqcup \bigcirc) = 0$ (equivalently, $S^2 \times S^1 = \mathbf{\Sigma}(S^3, \bigcirc \sqcup \bigcirc)$).

We have the material in place to show that this class of branch sets has thin Khovanov homology, a result that is due to Lee by virtue of the fact that non-split two-bridge links are alternating [29]. Since both $\tau(\frac{1}{0})$ and $\tau(1)$ are the trivial knot, applying Lemma 4.10 we have that

$$\mathbb{F} \cong H_*(\widetilde{\mathrm{Kh}}(\tau(0)) \to \mathbb{F}).$$

Recall that $\widetilde{\mathrm{Kh}}(\tau(0)) \cong \mathbb{F} \oplus \mathbb{F}$ as a relatively $\mathbb{Z}$-graded group. Now it follows that the branch sets corresponding to positive integer surgery have Khovanov homology

$$\widetilde{\mathrm{Kh}}(\tau(n)) \cong H_*(\widetilde{\mathrm{Kh}}(\tau(0)) \to \mathbb{F}[\mathbb{Z}/bZ]) \cong \mathbb{F}^n,$$

hence $w(\tau(n))$ is thin for all $n \neq 0$ (see also Lemma 4.17).

Without loss of generality, we consider $\widetilde{\mathrm{Kh}}(\tau(\frac{p}{q}))$ for $\frac{p}{q} > 0$. In fact, $\tau[0, a_2, a_3, \ldots, a_r] \simeq \tau[a_3, \ldots, a_r]$, so we need only consider $\frac{p}{q} \geq 1$. Now it is a quick application of Proposition 4.20 to see that $\tau(\frac{p}{q})$ is a thin link, for all $\frac{p}{q} \neq 0$, since $\tau(n)$ is thin for all $n \neq 0$. $\square$

In constructing two-bridge links in this way, we recover Schubert's normal form for this class [48]. Note that this proof that two-bridge knots are thin may be viewed as an adaptation of the proof that quasi-alternating knots are thin, due to Manolescu and Ozsváth [31], applied to two-bridge knots – a class of knots that are alternating, hence quasi-alternating.

5.2. **Width bounds for finite fillings.** Our main goal of this section is to prove an analogous statement in the case of manifolds with finite fundamental group.

Theorem 5.2. *If $\mathbf{\Sigma}(S^3, L)$ has finite fundamental group then $w(L)$ is at most 2.*

Proof. As a consequence of orbifold geometrization (for cyclic type in this setting), $|\pi_1(\mathbf{\Sigma}(S^3, L))| < \infty$ is equivalent to $\mathbf{\Sigma}(S^3, L)$ admitting elliptic geometry (see Thurston [54], Boileau and Porti [7]). Note that by Theorem 5.1 any lens space $\mathbf{\Sigma}(S^3, L)$ satisfies the bound of Theorem 5.2 since $w(L) = 1$ in this case. According to Scott, the remaining manifolds with elliptic geometry are Seifert fibered with 3 singular fibres and base orbifold S^2, and fall into two classes: either $S^2(2, 2, n)$ for $n > 1$ or $S^2(2, 3, n)$ for $n = 3, 4, 5$ [49]. The manifolds in each class may be constructed by considering fillings of Seifert fibered knot manifolds with base orbifold $D^2(2, 2)$ (the twisted I-bundle over the Klein bottle) and $D^2(2, 3)$ (the trefoil complement), respectively (see Heil [19], Montesinos [34]).

To any Seifert fibered space Y with base orbifold $S^3(p, q, r)$, Montesinos constructs a strong inversion so that $Y \cong \mathbf{\Sigma}(S^3, L)$ (see Proposition 3.6). In the case where Y has finite fundamental group, a result of Boileau and Otal says that this involution is unique [6]. As a consequence, it suffices to construct the family of manifolds in each class in such a way that the branch set is made explicit.

When filling the complement of the trefoil we appeal to Proposition 4.9: the branch set associated to filling any torus knot in S^3 has width at most 2. To complete the proof then, we are left to consider the case of filling the twisted I-bundle over the Klein bottle, M.

When considered with Seifert structure $D^2(2, 2)$, this manifold has the property that $\Delta(\varphi, \lambda_M) = 1$, where φ is a regular fibre in the boundary. Note that $M(\lambda_M)$ must be $S^2 \times S^1$, and $M(n\varphi + \lambda_M)$ is a lens space for all $n \neq 0$ by work of Heil [19]. By fixing a representative for the associated quotient tangle compatible with the basis for surgery (φ, λ_M) it follows that $w(\tau(n)) = 1$ for all $n \neq 0$, and $w(\tau(0)) = 2$. Now resolving the terminal crossing in $\tau(\frac{p}{q})$ we have, for $\frac{p}{q} \geq 0$,

$$\begin{aligned}
\det(\tau(\tfrac{p}{q})) &= |H_1(M(p\varphi + q\lambda_M); \mathbb{Z})| \\
&= c_M \Delta(p\varphi + q\lambda_M, \lambda_M) \\
&= c_M |p\varphi \cdot \lambda_M| \\
&= c_M |(p_0 + p_1)\varphi \cdot \lambda_M + (q_0 + q_1)\lambda_M \cdot \lambda_M| \\
&= c_M |(p_0\varphi + q_0\lambda_M) \cdot \lambda_M| + c_M |(p_1\varphi + q_1\lambda_M) \cdot \lambda_M| \\
&= c_M \Delta(p_0\varphi + q_0\lambda_M, \lambda_M) + c_M \Delta(p_1\varphi + q_1\lambda_M, \lambda_M) \\
&= |H_1(M(p_0\varphi + q_0\lambda_M); \mathbb{Z})| + |H_1(M(p_1\varphi + q_1\lambda_M); \mathbb{Z})| \\
&= \det(\tau(\tfrac{p_0}{q_0})) + \det(\tau(\tfrac{p_1}{q_1}))
\end{aligned}$$

in terms of (φ, λ_M).

This is enough to obtain the result, proceeding as in the proof of Proposition 4.9 (by way of Proposition 4.20), working with a tangle compatible with the basis (φ, λ_M) in place of the preferred basis (μ, λ). ☐

This result should be compared with [40, Proposition 2.3]: Ozsváth and Szabó show that manifolds with elliptic geometry are all L-spaces.

5.3. **Width obstructions.** We are now in a position to assemble the material developed to this point into obstructions to exceptional surgeries.

Theorem 5.3. *Let M be a simple, strongly invertible knot manifold with associated quotient tangle $T=(B^3,\tau)$ compatible with some basis (α,β) in ∂M. Then $w(\tau(\frac{p}{q}))>1$ implies that $M(p\alpha+q\beta)$ is not a lens space, and $w(\tau(\frac{p}{q}))>2$ implies that $M(p\alpha+q\beta)$ has infinite fundamental group.*

Proof. For $w>1$ the statement follows from Theorem 5.1; for $w>2$ the statement follows from Theorem 5.2. □

Our aim is to show that this is an effective obstruction by applying the results of Section 4, and in particular the stability of Lemma 4.10. For this purpose we restrict to strongly invertible knots in S^3, though in practice results may be obtained more generally (see Section 6.5, for example).

Let $T=(B^3,\tau)$ be the preferred representative for the tangle associated to a strongly invertible knot in S^3. Recall that $\tau(\frac{1}{0})$ is the trivial knot, and

$$\widetilde{\mathrm{Kh}}(\tau(m+n))\cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m))\to\mathbb{F}[\mathbb{Z}/n\mathbb{Z}]\right)$$

for some explicit identification

$$\mathbb{F}[\mathbb{Z}/n\mathbb{Z}]\cong\mathbb{F}[q]/q^n\cong\widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x,y][0,q]$$

as a graded $\mathbb{F}$-vector space. Here, $n>0$ and the fixed grading shift depends on the tangle and the integer m (c.f. Lemma 4.10). If $\widetilde{\mathrm{Kh}}(\tau(m))\cong\mathbb{F}^{b_1}\oplus\cdots\oplus\mathbb{F}^{b_k}$ as a relatively $\mathbb{Z}$-graded $\mathbb{F}$-vector space, so that $w(\tau(m))=k$, then the graded vector space $\mathbb{F}[\mathbb{Z}/n\mathbb{Z}]$ is added to some fixed (relative) grading δ^+ for $1\le\delta^+\le k+1$.

In the situation that the width decays (c.f Definition 4.24), $\delta^+=2$, and in the situation that width expands, $\delta^+=k+1$ (c.f. Definition 4.23). If the width neither decays nor expands then the tangle will be referred to as width stable.

Definition 5.4. *The tangle $T=(B^3,\tau)$ is generic if it is width stable, or if the width decays (respectively, expands) then it is decay generic as in Definition 4.24 (respectively expansion generic as in Definition 4.23).*

A stronger form of genericity exists and will be useful in application.

Proposition 5.5. *If for each δ-grading supporting a non-trivial group there is a q-grading so that $\operatorname{rk}\widetilde{\mathrm{Kh}}^{\delta}(\tau(m))>\operatorname{rk}\widetilde{\mathrm{Kh}}^{\delta}_q(\tau(m))>1$, then the associated quotient tangle is generic.*

Proof. This is immediate from Lemma 4.10: since the graded vector space $\mathbb{F}[\mathbb{Z}/n\mathbb{Z}]$ has a unique generator in each secondary grading q, the condition $\operatorname{rk}\widetilde{\mathrm{Kh}}^{\delta}_q(\tau(m))>1$ ensures that $b_\delta\neq 0$ in

$\widetilde{\mathrm{Kh}}(\tau(m+n)) \cong \bigoplus_{\delta=1}^{k} \mathbb{F}^{b_\delta}$, for all n. As a result, the tangle is either width stable, or it is expansion generic as a result of $\operatorname{rk} \widetilde{\mathrm{Kh}}^{\delta}(\tau(m)) > \operatorname{rk} \widetilde{\mathrm{Kh}}^{\delta}_{q}(\tau(m))$. □

Our main results then are the following:

Theorem 5.6. *Let $K \hookrightarrow S^3$ be strongly invertible with generic preferred associated quotient tangle. Then $w_{\min} > 1$ implies that K does not admit lens space surgeries. Moreover, determining $w_{\min}$ is a finite check by stability.*

Proof. This is an application of Theorem 5.1, together with the fact that $w_{\min}$ is determined as a result of Lemma 4.10 and provides a lower bound for $w(\tau(\frac{p}{q}))$ according to the results of Section 4. □

Theorem 5.7. *Let $K \hookrightarrow S^3$ be strongly invertible with generic preferred associated quotient tangle. Then $w_{\min} > 2$ implies that K does not admit finite fillings. Moreover, determining $w_{\min}$ is a finite check by stability.*

Proof. Similarly, this is an application of Theorem 5.2, together with the fact that $w_{\min}$ is determined as a result of Lemma 4.10 and provides a lower bound for $w(\tau(\frac{p}{q}))$ according to the results of Section 4. □

Remark 5.8. *In practice, one group $\widetilde{\mathrm{Kh}}(\tau(m))$ is often enough to determine $w_{\min}$ and apply these obstructions.*

In the absence of the genericity hypothesis, the width is still a useful obstruction: In light of the cyclic surgery theorem [10], it is enough to check the integer fillings of K when the question of lens space surgeries is of interest. Similarly, in the case of finite fillings only the integer and half-integer surgeries need to be considered in light of work of Boyer and Zhang, whenever the complement admits a hyperbolic structure [9]. In practice however, genericity is easy to check and seems to be the rule and not the exception. In the generic setting (see examples given below), it is particularly interesting that Khovanov homology is able to give useful surgery obstructions, without relying on these powerful theorems.

6. Examples

6.1. **A first example: the figure eight.** It is well known that the figure eight knot $K = 4_1$ does not admit lens space surgeries. In fact, Thurston [53] classified the non-hyperbolic fillings of $S^3 \smallsetminus \nu(K)$ and showed that they all have infinite fundamental group. That K does not admit lens space surgeries has been reproved using the machinery of instanton Floer homology [26, 27], essential laminations [11], character varieties [52] and most recently, Heegaard-Floer homology [40]. As a first example of the width obstructions developed here, we show that Khovanov homology detects that K does not admit finite fillings.

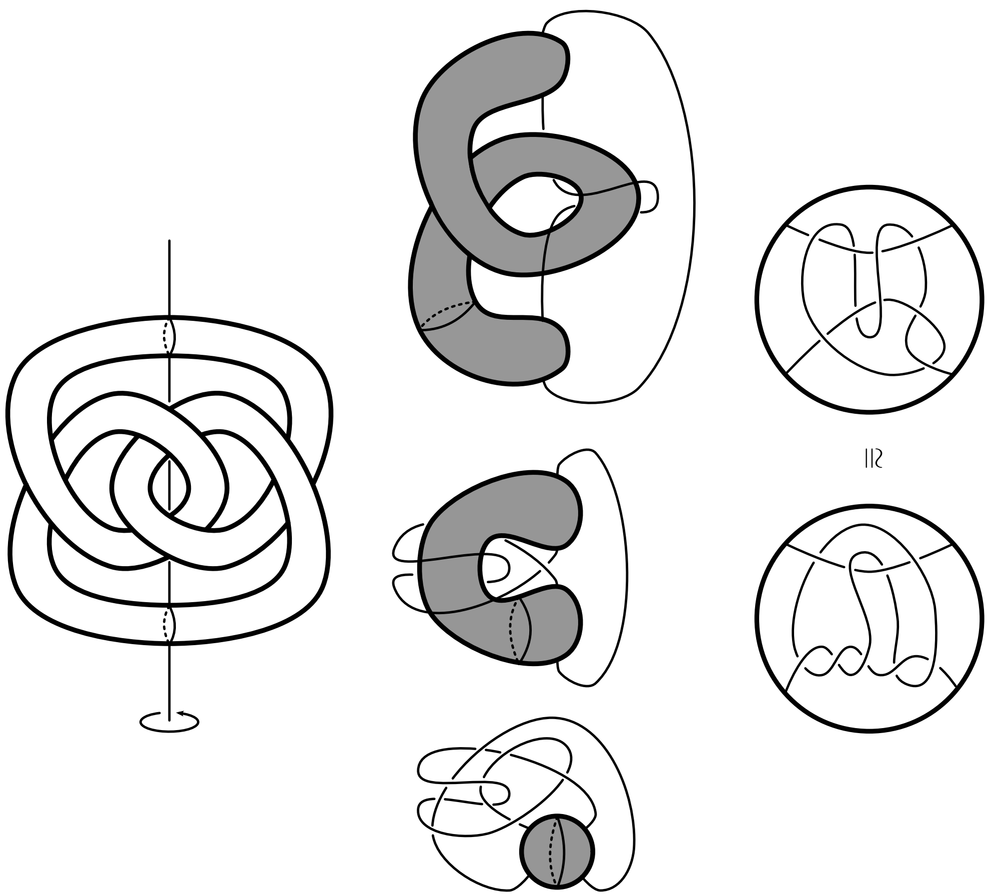


FIGURE 11. The strong inversion on the figure eight (left); isotopy of a fundamental domain (centre); and two representatives of the associated quotient tangle (right).

K is a strongly invertible knot, and this symmetry is shown in Figure 11 together with the associated quotient tangle. We have given two equivalent views of the associated quotient tangle. The first of these shows that the branch sets for integer surgeries may be expressed as closed 3-braids. For

$$\beta_n = \sigma_1^{-1}\sigma_2^{-2}\sigma_1^{-2}\sigma_2^{-2}\sigma_1^{-2}\sigma_2^{-4+n}$$

we have that $\tau(n) \simeq \overline{\beta_n}$, the closure of β_n. The Khovanov homology groups $\widetilde{\mathrm{Kh}}(\tau(-1))$, $\widetilde{\mathrm{Kh}}(\tau(0))$ and $\widetilde{\mathrm{Kh}}(\tau(+1))$ are given in Figure 12 (note in particular that $\chi(\widetilde{\mathrm{Kh}}(\tau(0))) = \det(\tau(0)) = 0$). Notice that $w_{\min} = 2$ and that the tangle is decay generic. It follows at once that K does not admit lens space surgeries applying Theorem 5.6, and it seems worth pointing out that this result could have been inferred simply by inspection of the single Khovanov homology group $\widetilde{\mathrm{Kh}}(\tau(0))$.

More generally, we may use Lemma 4.10 to calculate:

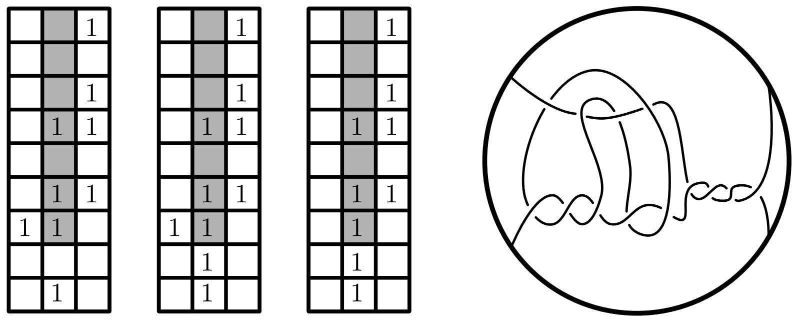


FIGURE 12. The preferred representative for the associated quotient tangle $T = (B^3, \tau)$ of the figure eight, and the reduced Khovanov homology groups $\widetilde{\mathrm{Kh}}(\tau(-1))$, $\widetilde{\mathrm{Kh}}(\tau(0))$ and $\widetilde{\mathrm{Kh}}(\tau(1))$ (from left to right). The δ^+ grading has been highlighted, in accordance with Lemma 4.10 setting $m = 0$.

Proposition 6.1.

$$\widetilde{\mathrm{Kh}}(\tau(n)) \cong \begin{cases} \mathbb{F}^{4+n} \oplus \mathbb{F}^4 & n > 0 \\ \mathbb{F} \oplus \mathbb{F}^5 \oplus \mathbb{F}^4 & n = 0 \\ \mathbb{F}^{|n|} \oplus \mathbb{F}^4 \oplus \mathbb{F}^4 & n < 0 \end{cases}$$

Proof. The grading δ^+ is identified in Figure 12. By calculating that $\widetilde{\mathrm{Kh}}(\tau(-2)) \cong \mathbb{F}^2 \oplus \mathbb{F}^4 \oplus \mathbb{F}^4$, Lemma 4.10, together with the groups

$$\begin{aligned} \widetilde{\mathrm{Kh}}(\tau(-1)) &\cong \mathbb{F} \oplus \mathbb{F}^4 \oplus \mathbb{F}^4 \\ \widetilde{\mathrm{Kh}}(\tau(0)) &\cong \mathbb{F} \oplus \mathbb{F}^5 \oplus \mathbb{F}^4 \\ \widetilde{\mathrm{Kh}}(\tau(1)) &\cong \qquad \mathbb{F}^5 \oplus \mathbb{F}^4 \end{aligned}$$

forces the result. □

In fact, we have enough to recover Thurston's result:

Theorem 6.2. *Khovanov homology detects that the figure eight admits no finite fillings.*

Proof. First notice that $w(\tau(n)) = 3$ for $n \leq 0$. As a result, a finite filling cannot arise by negative surgery on the figure eight by Theorem 5.2. However, since the figure eight knot is amphicheiral, the same must be true for positive surgeries. □

6.2. **Some pretzel knots that do not admit finite fillings.** According to Mattman [32], it is unknown if the $(-2, p, q)$-pretzel knots admit fillings with finite fundamental group for $q \geq p \geq 5$. When $p = q = 5$ we have the following.

Theorem 6.3. *The $(-2, 5, 5)$-pretzel knot does not admit finite fillings.*

Proof. We begin by noting that the $(-2,5,5)$-pretzel knot, K_5, is strongly invertible in two ways as indicated in Figure 13. We will make use of the inversion indicated by the solid vertical line; the

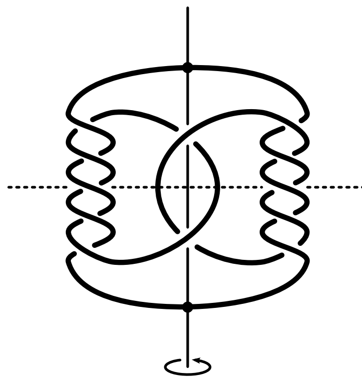


FIGURE 13. Two strong inversions on the $(-2,5,5)$-pretzel knot.

associated quotient tangle is calculated in Figure 14. Notice that the associated quotient tangle in

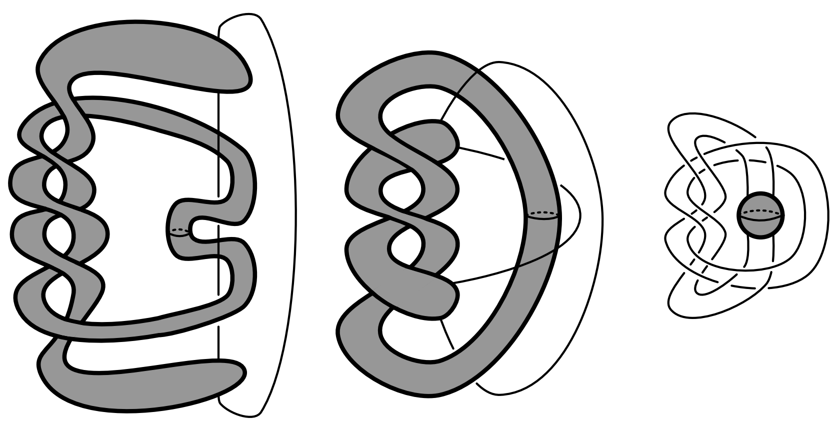


FIGURE 14. Isotopy of the fundamental domain for a strong inversion on the $(-2,5,5)$-pretzel knot. Notice that the resulting tangle has the property that integer closures are representable by closed 4-braids.

this case gives rise to an obvious collection of 4-braids, the closures of which give the branch sets for integer fillings. Setting

$$\beta_n = \sigma_2^{-1}\sigma_3^{-1}\sigma_1\sigma_2\sigma_1^{14+n}\sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}(\sigma_2^{-1}\sigma_3^{-1}\sigma_1^{-1}\sigma_2^{-1})^3$$

we have $\tau(n) = \overline{\beta_n}$ by verifying that $\widetilde{\mathrm{Kh}}(\tau(0)) \cong \mathbb{F}^{16} \oplus \mathbb{F}^{20} \oplus \mathbb{F}^4$ so that $\det(\tau(0)) = 0$. The homologies of $\tau(n)$ for $n = -18, -17, -16, -15, -14$ are given in Figure 15. This data is enough to

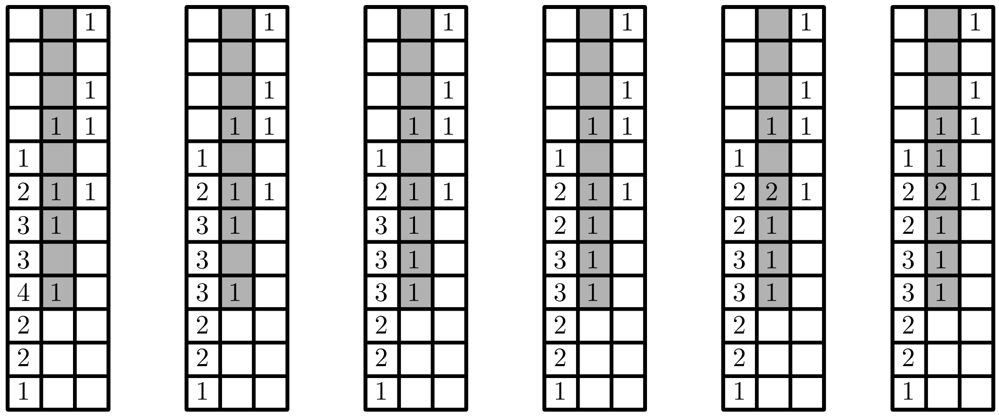


FIGURE 15. $\widetilde{\mathrm{Kh}}(\tau(n))$ for $n = -18, -17, -16, -15, -14$ (from left to right).

infer that

$$\widetilde{\mathrm{Kh}}(\tau(n)) \cong \begin{cases} \mathbb{F}^{-n} \oplus \mathbb{F}^4 \oplus \mathbb{F}^4 & n < -16 \\ \mathbb{F}^{17} \oplus \mathbb{F}^5 \oplus \mathbb{F}^4 & n < -16 \\ \mathbb{F}^{16} \oplus \mathbb{F}^{20+n} \oplus \mathbb{F}^4 & n > -16 \end{cases}$$

as relatively $\mathbb{Z}$-graded groups. In particular, $w_{\min} = w_{\max} = 3$ so the associated quotient tangle is generic. The result now follows from Theorem 5.7. □

Considering the same involution on the $(-2, p, p)$-pretzel knot K_p for all $p \geq 5$ we have that, in terms of the preferred associated quotient tangle, $\tau_p(n) = \overline{\beta_{n,p}}$ where

$$\beta_{n,p} = \sigma_2^{-1}\sigma_3^{-1}\sigma_1\sigma_2\sigma_1^{24+2p+n}\sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}(\sigma_2^{-1}\sigma_3^{-1}\sigma_1^{-1}\sigma_2^{-1})^{p-2}$$

so that $S^3_n(K_p) \cong \mathbf{\Sigma}(S^3, \tau_p(n))$. Notice that, aside from the exponent $24 + 2p + n$ corresponding to the surgery coefficient, this expression changes only the number of double-strand full-twists in the associated quotient tangle (see Figure 14). From this expression, we calculate $\widetilde{\mathrm{Kh}}(\tau_p(24 + 2p))$ for p odd in the range $5 \leq p \leq 31$.These calculations yield generic tangles in each case, with $w(\tau_p(24 + 2p)) = p - 2$, from which we conclude:

Theorem 6.4. *The $(-2, p, p)$-pretzel knots do not admit finite fillings for $5 \leq p \leq 31$.*

It seems reasonable to conjecture that $w = p - 2$ for the branch sets associated to surgery on K_p, for any $p \geq 5$, so that Khovanov homology obstructs finite fillings on this class of knots. We do not pursue this here, since the result may be shown by other means. Indeed, it is possible to use obstructions from Heegaard-Floer homology to rule out L-space surgeries by considering the Alexander polynomials of the $(-2, p, p)$-pretzel knots, as pointed out to the author by M. Hedden. This has been carried out very recently by Ichihara and Jong completing Mattman's classification of Montesinos knots admitting finite fillings [22]. Since then the result has received a different treatment by Futer, Ishikawa, Kabaya, Mattman and Shimokawa [13].

We remark that Mattman's classification [32] using character variety methods illustrates some subtleties. Indeed, the $(-2,3,q)$-pretzel knots admit L-space surgeries for all $q \geq 3$ (see [40]). Despite this fact however, Mattman shows that for $q > 9$ none of these manifolds can have finite fundamental group. On the other hand, for the $(-2,p,p)$-pretzel knots the character variety methods of Mattman were inconclusive, but this is precisely the setting in which Heegaard-Floer homology – and, as seen here, Khovanov homology – obstructs finite fillings.

6.3. **Khovanov homology obstructions and Heegaard-Floer homology.** In light of the discussion above, it is natural to put the obstructions from Khovanov homology in contrast with those coming from Heegaard-Floer homology. The latter theory gives very stringent restrictions for the knot Floer homology of a knot admitting an L-space surgery [40]. Ozsváth and Szabó give the following quickly implemented obstruction from the Alexander polynomial.

Theorem 6.5 (Ozsváth-Szabó [40, Corollary 1.3])**.** *A knot $K \hookrightarrow S^3$ for which $S^3_n(K)$ is an L-space (for some $n \in \mathbb{Z}$) has Alexander polynomial of the form*

$$\Delta_K(t) = (-1)^k + \sum_{j=1}^{k} (-1)^{k-j}(t^{-n_j} + t^{n_j})$$

for some increasing sequence of integers $0 < n_1 < n_2 < \cdots < n_k$.

Manifolds elliptic geometry are known to be L-spaces [40, Proposition 2.3], so this gives a useful obstruction to finite fillings in the present context. However, the criteria given in Theorem 6.5 can fail. For example,

$$\Delta_K(t) = t^{-3} - t^{-2} + t^{-1} - 1 + t - t^2 + t^3$$

where K is the 14 crossing, non-alternating knot shown in Figure 16. Since this is a strongly

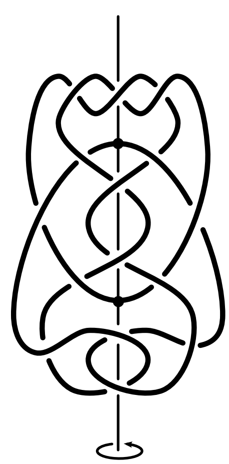


FIGURE 16. The strongly invertible knot $K = 14^n_{11893}$ has Alexander polynomial $\Delta_K(t) = t^{-3} - t^{-2} + t^{-1} - 1 + t - t^2 + t^3$.

invertible knot, we are in a position to apply width obstructions from Khovanov homology. The

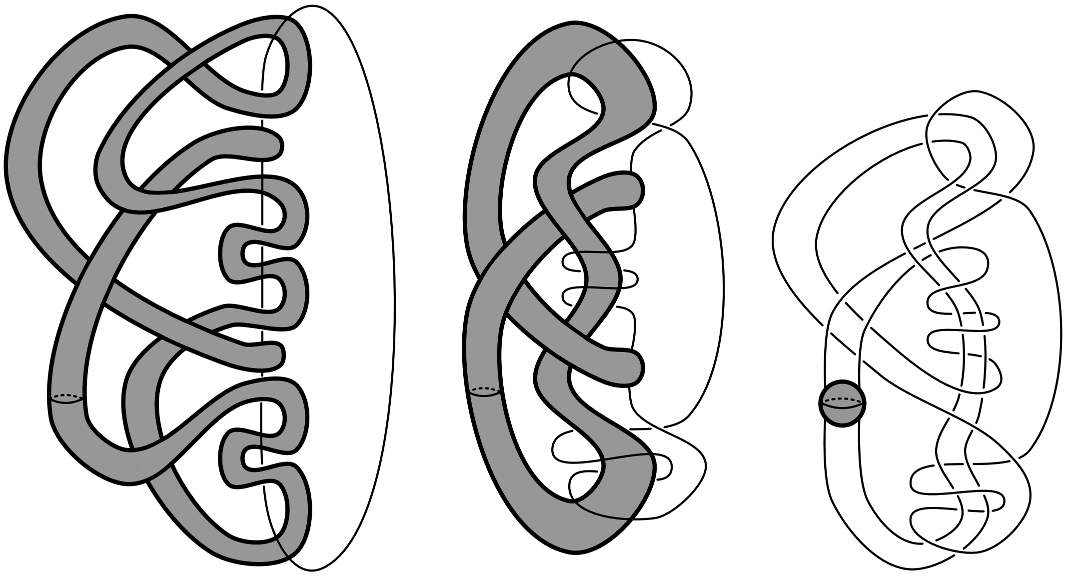


FIGURE 17. Isotopy of a fundamental domain for the involution on the complement of 14^n_{11893}.

associated quotient tangle is determined in Figure 17: notice that by construction the trivial knot $\tau(\frac{1}{0})$ is obtained by connecting the endpoints of the arcs of τ with two horizontal arcs inside the small sphere shown. Therefore, without knowing the framing, we can be sure that the branch sets for integer surgeries result from adding vertical half-twists inside the sphere, as shown in Figure 18.

By inspection of the resulting group in Figure 18, the associated quotient tangle is generic since for each δ-grading supporting a non-trivial group there is at least one q for which $\operatorname{rk}\mathrm{Kh}^{\delta}_{q}(\tau(n)) > 1$ (see Proposition 5.5). The width is at least 4, for all n, and as a result we conclude:

Theorem 6.6. 14^n_{11893} *does not admit finite fillings; one Khovanov homology group suffices.*

In this setting, by switching the circled crossing of Figure 18 from positive to negative, we can determine that

$$\widetilde{\mathrm{Kh}}(\tau(-9)) \cong \mathbb{F}^{20} \oplus \mathbb{F}^{36} \oplus \mathbb{F}^{41} \oplus \mathbb{F}^{16}$$
$$\widetilde{\mathrm{Kh}}(\tau(-7)) \cong \mathbb{F}^{20} \oplus \mathbb{F}^{36} \oplus \mathbb{F}^{39} \oplus \mathbb{F}^{16}$$

so that $w_{\min} = w_{\max} = 4$ and as T is generic, this determines the width of the branch set for *any* surgery on K.

While it is possible that the full knot Floer homology of K obstructs L-space surgeries, this example shows that in certain settings the Khovanov homology obstructions may be more convenient from a computational standpoint when the question of finite fillings is of interest. Further, these obstructions may allow one to rule out finite fillings among L-spaces, a distinction that can be subtle.

			1
			2
			2
		2	3
		4	3
		5	2
	2	7	2
	4	7	1
	5	6	
1	6	5	
2	7	3	
3	5		
3	4		
4	3		
3			
2			
2			

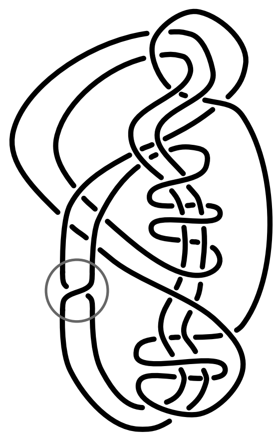


FIGURE 18. The branch set for some some integer surgery $S^3_n(K)$. Note that $\widetilde{\mathrm{Kh}}(\tau(n)) \cong \mathbb{F}^{20} \oplus \mathbb{F}^{36} \oplus \mathbb{F}^{39} \oplus \mathbb{F}^{16}$ so that $\chi = 59 - 52 = 7$ and $n = \pm 7$.

6.4. **A characterization of the trivial knot.** We turn now to an observation regarding Khovanov homology and non-trivial, strongly invertible knots in S^3.

Theorem 6.7. *Let K be a strongly invertible knot in S^3 with preferred associated quotient tangle $T = (B^3, \tau)$. Then $\widetilde{\mathrm{Kh}}(\tau(n))$ is thin for every non-zero integer n if and only if K is the trivial knot.*

First, we prove the following.

Proposition 6.8. *$S^3_n(K)$ is an L-space for every non-zero integer n if and only if K is the trivial knot.*

Proof. If K is the trivial knot, then $S^3_n(K)$ is a lens space for $n \neq 0$ and the result follows. We treat the converse.

Suppose $S^3_n(K)$ is an L-space for every non-zero integer n. Then in particular $S^3_{+1}(K)$ is an L-space and in follows that K has genus at most 1 [28, Corollary 8.5] (see also [17, 46]). In this setting, if K is non-trivial, work of Ghiggini [14] implies that K must be the right-hand trefoil (see also [18, Proposition 5]).

It is well known that -1-surgery on the right-hand trefoil yields the same manifold as $+1$-surgery on the figure eight (see for example Rolfsen [47, Chapter 9]). Since the latter knot is alternating but not torus, applying [40, Theorem 1.5] shows that the manifold resulting from $+1$-surgery on the figure eight knot cannot be an L-space. (Equivalently, it can be seen by direct computation via

the mapping cone formula for integer surgeries [42] that -1-surgery on the right-hand trefoil is not an L-space. Note also that this calculation was originally given in [39].)

As a result, K has genus 0 and must be the trivial knot. □

Proof of Theorem 6.7. If K is the trivial knot, then $\tau(n)$ is a two-bridge link, and $\widetilde{\mathrm{Kh}}(\tau(n))$ is thin for $n \neq 0$ (c.f. Theorem 5.1). We treat the converse.

Ozsváth and Szabó show that there is a spectral sequence with E_2 term given by the reduced Khovanov homology of the mirror of L, converging to $\widehat{\mathrm{HF}}(\mathbf{\Sigma}(S^3, L))$ [41]. As a result [41, Corollary 1.2], we have the following inequalities:

$$|H_1(\mathbf{\Sigma}(S^3, L); \mathbb{Z})| \leq \mathrm{rk}\, \widehat{\mathrm{HF}}(\mathbf{\Sigma}(S^3, L)) \leq \mathrm{rk}\, \widetilde{\mathrm{Kh}}(L)$$

Further, whenever $\widetilde{\mathrm{Kh}}(L)$ is thin $\mathbf{\Sigma}(S^3, L)$ is an L-space (see Proposition 4.2).

Now suppose that $\widetilde{\mathrm{Kh}}(\tau(n))$ is thin for every non-zero integer n. Then from the discussion above $\mathbf{\Sigma}(S^3, \tau(n)) \cong S^3_n(K)$ is an L-space for $n \neq 0$. If K is non-trivial, this contradicts Proposition 6.8. □

Remark 6.9. *Equivalently, Theorem 6.7 follows from Lemma 4.17.*

Using the symmetry group of the knot it is possible to determine when a knot is not strongly invertible. As a result, Khovanov homology may be used to detect the trivial knot in the following sense: since the trivial knot is strongly invertible, Khovanov homology, together with the symmetry group of the knot, detects the trivial knot via Theorem 6.7. Note that Lemma 4.10 ensures that the minimal width of $\widetilde{\mathrm{Kh}}(\tau(n))$ is determined on a finite collection of integers.

It is possible to show that the Khovanov homology of certain satellites of a knot, including the $(2,1)$-cable, detect the trivial knot [18], and this fact follows from a statement that is essentially equivalent to Proposition 6.8. In the present setting, it is the Khovanov homology of the branch set associated to a filling of the knot in question that detects the unknot. In light of the relationship between Heegaard-Floer homology and Khovanov homology by way of two-fold branched covers it is interesting to recall that knot Floer homology, which is closely tied to the Heegaard-Floer homology of surgeries on a knot, detects the trivial knot [38]. Here, Khovanov homology detects the trivial knot among knots whose complements are branched covers of tangles.

6.5. **Knots in the Poincaré sphere: a final example.** In application of these surgery obstructions, the requirement that the knot be strongly invertible seems restrictive. However, while such an involution is required, we remark that the obstructions presented may be applied in a broader context, beyond knots in the S^3. As illustration, we study a particular example of surgery on a strongly invertible knot in the Poincaré homology sphere, Y. Dehn surgery on knots in this manifold have been considered by Tange in the context of the Berge conjecture [51].

Recall that $Y \cong \mathbf{\Sigma}(S^3, B)$ where B is the knot 10_{124}, the $(-2,3,5)$-pretzel. Consider the knot $K \hookrightarrow Y$ given by the lift $\widetilde{\gamma} = K$ where γ is the arc illustrated in Figure 19 with endpoints on the

FIGURE 19. The branch set B (the knot 10_{124}) and the arc γ giving rise to $\widetilde{\gamma} = K$ in the two-fold branched cover $Y = \mathbf{\Sigma}(S^3, B)$ (the Poincaré sphere). The preferred associated quotient tangle is shown on the right. Note that $\tau(\frac{1}{0}) \simeq B$ and $\widetilde{\mathrm{Kh}}(\tau(0)) \cong \mathbb{F}^{80} \oplus \mathbb{F}^{176} \oplus \mathbb{F}^{180} \oplus \mathbb{F}^{84}$ so that $\det(\tau(0)) = 0$.

branch set B. Note that $K \hookrightarrow Y$ is strongly invertible (by construction), and that $M = Y \smallsetminus \nu(K)$ is a simple, strongly invertible knot manifold.

	1
	1
1	1
1	1
1	

Since $H_1(Y;\mathbb{Z}) \cong 0$, there is a preferred longitudinal slope λ in ∂M so that $H_1(M(\lambda);\mathbb{Z}) \cong \mathbb{Z}$ and $\Delta(\mu, \lambda) = 1$. As a result, as in the case a of a knot complement in S^3, $M \cong \mathbf{\Sigma}(B^3, \tau)$ where we fix the preferred representative $T = (B^3, \tau)$ of associated quotient tangle. This tangle is illustrated in Figure 19; notice that $\tau(\frac{1}{0}) \simeq B$ is obtained by filling with the tangle $(B^3,)()$ (thus, a branch set for the trivial surgery on K) and $Y_0(K) \cong \mathbf{\Sigma}(S^3, \tau(0))$ where $\tau(0)$ is obtained by filling with $(B^3, \asymp)$. In analysing the homology $\widetilde{\mathrm{Kh}}(\tau(\frac{p}{q}))$ for the branch sets associated to $Y_{p/q}(K)$, first recall that $\widetilde{\mathrm{Kh}}(\tau(\frac{1}{0})) \cong \mathbb{F}^3 \oplus \mathbb{F}^4$ as a singly graded group (the bi-graded group is illustrated on the right). As a result, we do not have a general form of stability as in Lemma 4.10, a priori. However, it will make sense to consider the groups $\widetilde{\mathrm{Kh}}(\tau(m \pm 1))$ for a fixed integer m. For example, when $m = 0$ we have that

$$\begin{aligned} \widetilde{\mathrm{Kh}}(\tau(+1)) &\cong \mathbb{F}^{80} \oplus \mathbb{F}^{176} \oplus \mathbb{F}^{183} \oplus \mathbb{F}^{88} \\ \widetilde{\mathrm{Kh}}(\tau(0)) &\cong \mathbb{F}^{80} \oplus \mathbb{F}^{176} \oplus \mathbb{F}^{180} \oplus \mathbb{F}^{84} \\ \widetilde{\mathrm{Kh}}(\tau(-1)) &\cong \mathbb{F}^{80} \oplus \mathbb{F}^{176} \oplus \mathbb{F}^{177} \oplus \mathbb{F}^{80} \end{aligned}$$

as relatively $\mathbb{Z}$-graded groups (which verifies in particular that $\det(\tau(0)) = 0$ and $\det(\tau(\pm 1)) = 1$, as claimed). Notice that this forces each of

$$\widetilde{\mathrm{Kh}}(\tau(0)) \cong H_* \left(\widetilde{\mathrm{Kh}}(\tau(-1)) \xrightarrow{0} \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0})) \right)$$

and

$$\widetilde{\mathrm{Kh}}(\tau(+1)) \cong H_* \left(\widetilde{\mathrm{Kh}}(\tau(0)) \xrightarrow{0} \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0})) \right)$$

for dimension reasons (suppressing the grading shifts), since in each case the groups in (relative) grading 3 and 4 are increased by 3 and 4 respectively.

This behaviour should not be expected in general, though we do have that

$$\widetilde{\mathrm{Kh}}(\tau(m+1) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m)[-\tfrac{1}{2},\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}(\tfrac{1}{0})[-\tfrac{1}{2}(c_\tau+m),\tfrac{1}{2}(3c_\tau+3m+2)]\right),$$

and this mapping cone may be iterated as in the proof of Lemma 4.10. For example, the groups

$$\widetilde{\mathrm{Kh}}(\tau(-11)) \cong \mathbb{F}^{80}\oplus\mathbb{F}^{176}\oplus\mathbb{F}^{155}\oplus\mathbb{F}^{48}$$
$$\widetilde{\mathrm{Kh}}(\tau(-10)) \cong \mathbb{F}^{80}\oplus\mathbb{F}^{176}\oplus\mathbb{F}^{154}\oplus\mathbb{F}^{48}$$
$$\widetilde{\mathrm{Kh}}(\tau(-9)) \cong \mathbb{F}^{80}\oplus\mathbb{F}^{176}\oplus\mathbb{F}^{153}\oplus\mathbb{F}^{48}$$

are illustrate in Figure 20. When $m=-11,-10$, these groups illustrate the behaviour of the above mapping cone. Notice that the total rank decreases by one in each case. More generally, though differentials among the $\widetilde{\mathrm{Kh}}(\tau(\frac{1}{0}))[x,y][0,q]$ may be present, the groups still only occupy two fixed diagonals when $\widetilde{\mathrm{Kh}}(\tau(m+n))$ is viewed as a relatively graded group.

			1
			3
		1	5
		5	7
		11	8
	1	17	8
	5	22	7
	12	25	5
	19	24	3
1	25	21	1
4	29	15	
7	28	9	
10	24	4	
13	18	1	
13	10		
12	4		
10	1		
6			
3			
1			

			1
			3
		1	5
		5	7
		10	8
	1	17	8
	5	21	7
	12	25	5
	19	24	3
1	25	21	1
4	29	16	
7	28	9	
10	24	4	
13	18	1	
13	10		
12	4		
10	1		
6			
3			
1			

			1
			3
		1	5
		4	7
		10	8
	1	16	8
	5	21	7
	12	25	5
	19	24	3
1	25	22	1
4	29	16	
7	28	9	
10	24	4	
13	18	1	
13	10		
12	4		
10	1		
6			
3			
1			

FIGURE 20. The groups $\widetilde{\mathrm{Kh}}(\tau(-11))$, $\widetilde{\mathrm{Kh}}(\tau(-10))$ and $\widetilde{\mathrm{Kh}}(\tau(-9))$ from left to right. The change in each group (corresponding to a $+1$ surgery in the cover) is circled; the support of $\widetilde{\mathrm{Kh}}(\tau(\frac{1}{0}))\cong\mathbb{F}^3\oplus\mathbb{F}^4$ is shaded in grey so that $\widetilde{\mathrm{Kh}}(\tau(m+1))\cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m))\to(\mathbb{F}^3\oplus\mathbb{F}^4)\right)$.

We now analyse the behaviour of $w(\tau(n))$ for $n \in \mathbb{Z}$. First notice that

$$\widetilde{\mathrm{Kh}}(\tau(0) \cong \mathbb{F}^{80} \oplus \mathbb{F}^{176} \oplus \mathbb{F}^{180} \oplus \mathbb{F}^{84}$$

so that

$$\widetilde{\mathrm{Kh}}(\tau(1)) \cong H_* \left(\begin{tikzcd} \mathbb{F}^{80} & \mathbb{F}^{176} \arrow[dr, "0"] & \mathbb{F}^{180} \arrow[dr, "0"] & \mathbb{F}^{84} \\ & & \mathbb{F}^{3} & \mathbb{F}^{4} \end{tikzcd} \right)$$

by our calculations above. More generally, for $m > 0$

$$\widetilde{\mathrm{Kh}}(\tau(m+1) \cong H_* \left(\begin{tikzcd} \mathbb{F}^{b_1} & \mathbb{F}^{b_2} \arrow[dr] & \mathbb{F}^{b_3} \arrow[dr] & \mathbb{F}^{b_4} \\ & & \mathbb{F}^{3} & \mathbb{F}^{4} \end{tikzcd} \right)$$

by analysing the grading shifts as in the proof of Lemma 4.10. In particular, $b_i > 0$ for all $m > 0$ due to the shift by 1 in the secondary grading at each step (note that $b_1 = 80$, for all m).

Similarly, notice that

$$\widetilde{\mathrm{Kh}}(\tau(-1) \cong H_* \left(\begin{tikzcd} & \mathbb{F}^{3} \arrow[dr, hook] & \mathbb{F}^{4} \arrow[dr, hook] & \\ \mathbb{F}^{80} & \mathbb{F}^{176} & \mathbb{F}^{180} & \mathbb{F}^{84} \end{tikzcd} \right)$$

this time by resolving the single negative terminal crossing (corresponding to the -1-surgery in the cover). More generally, for $m < 0$

$$\widetilde{\mathrm{Kh}}(\tau(m-1) \cong H_* \left(\begin{tikzcd} & \mathbb{F}^{3} \arrow[dr] & \mathbb{F}^{4} \arrow[dr] & \\ \mathbb{F}^{b_1} & \mathbb{F}^{b_2} & \mathbb{F}^{b_3} & \mathbb{F}^{b_4} \end{tikzcd} \right)$$

by inspection of the grading shifts as in Lemma 4.10. Analysing the groups in Figure 20, we see that $b_1 = 80$ as before (for any m), while b_4 is necessarily non-trivial due to the shift by -1 in the secondary grading at each step.

As a result, we conclude that $w(\tau(n)) = 4$ for every $n \in \mathbb{Z}$. With this in hand, we may determine $w(\tau(\frac{p}{q}))$ for every $\frac{p}{q} \in \mathbb{Q}$: $w(\tau(\frac{p}{q}))$ is bounded above by 4 (proceeding as in Proposition 4.20) and bounded below by 4 (proceeding as in Proposition 4.22, since $w_{\min} = w_{\max} = 4$ in this case). Said another way, the function $w(\tau(-)) : \mathbb{Q} \to \mathbb{N}$ is constant, with value 4. As a result, applying Theorem 5.2 we conclude that $K \hookrightarrow Y$ does not admit finite fillings.

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Département de Mathématiques, Université du Québec à Montréal, Montréal Canada.

E-mail address: liam.watson@cirget.ca

URL: http://www.cirget.uqam.ca/~liam